9^{th} Math exercise 5.2

Q. 1 Maximize $f(x, y) = 2x + 5y$; subject to the constraints

$$2y - x \leq 8;\ x - y \leq 4;\ x \geq 0; y \geq 0$$

Solution:

The constraints are

$$2y - x \leqq 8$$
$$x - y \leqq 4$$
$$x \geq 0, y \geq 0$$

The sketch of inequality (i) is shown as shaded region in the figure 1 (a) by joining the points $\mathrm{A}(-8,0)$ and $\mathrm{B}(0,4)$ which are x and y intercepts respectively.

The sketch of inequality (ii) is shown as shaded region in the figure 1 (b) by joining the points $C(4,0)$ and $D(0,-4)$ which are x and y intercepts respectively.

The sketch of inequalities $x \geq 0, y \geq 0$ (iii) is shown as shaded region in the figure $1(\mathrm{c})$

Now the solution region of the given systems of inequalities is the intersection of the graphs indicated in the figure 1 (a), 1 (b), 1 (c) and is shown as shaded region OCEB in the figure 1 (d).

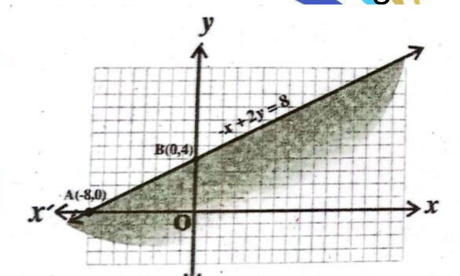

Fig.1(a)

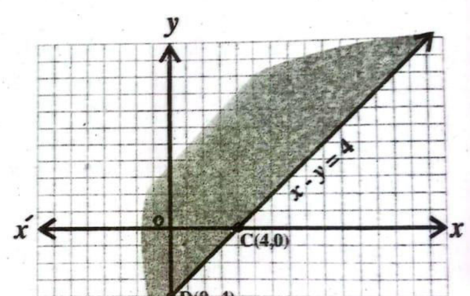

Fig.1(b)

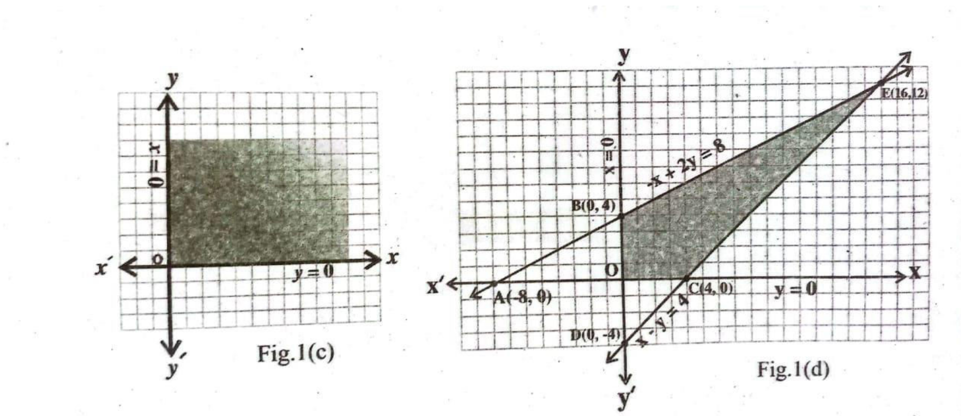

Fig.1(c) Fig.1(d)

So the corner points are $\mathrm{O}, \mathrm{C}, \mathrm{E}$ and B .
For the comer point E , solving the lines $2\mathrm{y}-x=8$ and $\mathrm{x}-\mathrm{y}=4$ simultaneously, we have the intersection point $\mathrm{E}(16,12)$. It can also be taken from the graph directly.
Thus comer points are $0(0,0), \mathrm{C}(4,0), \mathrm{E}(16,12)$ and $\mathrm{B}(0,4)$.
Now find the value of $f(x, y)$ at the comer points, we have

Corner points	$f(x, 1)=2x+5y$
$0(0,0)$	$f(0,0)=2(0)+5(0)=0$
$C(4,0)$	$f(4,0)=2(4)+5(0)=8$
$E(16,12)$	$f(16,12)=2(16)+5(12)=92$
$B(0,4)$	$f(0,4)=2(0)+5(4)=20$

We observe that the maximum value of $f(\mathrm{x}, \mathrm{y})$ is 92 at the comer point $\mathrm{E}(16,12)$.
Q. 2 Maximize $f(x, y)=x+3y$ subject to the constraints

$$2x+5y \leq 30; 5x+4y \leq 20; x \geq 0; y \geq 0$$

Solution:
The constraints are

$$2x+5y \leq 30$$
$$5x+4y \leq 20$$
$$x \geq 0, y \geq 0$$

The sketch of inequality (i) is shown as shaded region in the figure 2(a) by joining the points $\mathrm{A}(15,0)$ and $\mathrm{B}(0,6)$ which are x and y intercepts respectively

The sketch of in equalities (ii) is shown as shaded region in the figure 2(b) by joining the points $\mathrm{C}(4,0)$ and $\mathrm{D}(0,5)$ which are x and y intercepts respectively
The sketch of inequality (iii) is shown as shaded region in the figure (c)
Now the solution region of the given systems of inequalities is the intersection of the graphs indicated in the figure 2(a), 2(b), 2(c) and is shown as shaded region OCD in the figure 2(d).

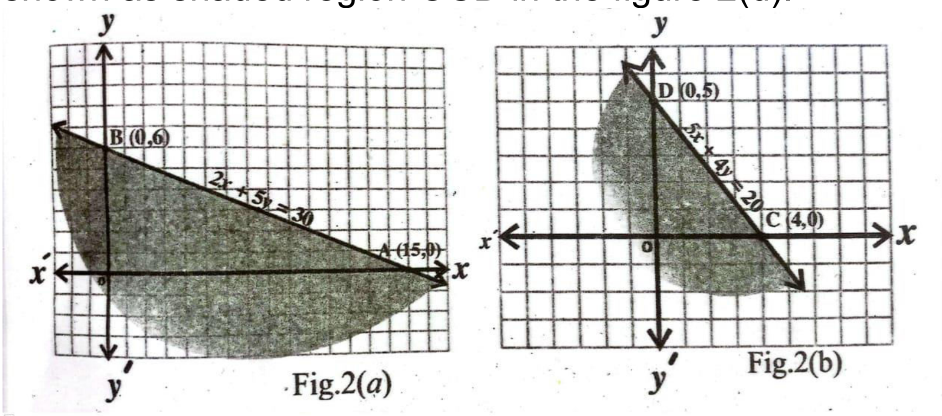

Fig.2(a) Fig.2(b)

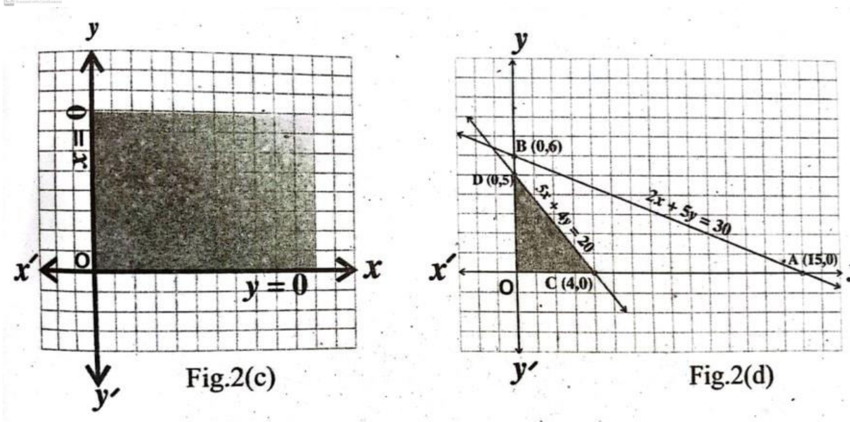

Fig.2(c) Fig.2(d)

y

Thus comer points are $\mathrm{O}(0,0), \mathrm{C}(4,0)$ and $\mathrm{D}(0,5)$.
Now find the value of $f(x, y)$ at the comer points, we have

Corner points	$f(x, y)=x+3y$
$\mathrm{O}(0,0)$	$f(0,0)=0+3(0)=0$
$\mathrm{C}(4,0)$	$f(4,0)=4+3(0)=4$
$\mathrm{D}(0,5)$	$f(0,5)=0+3(5)=15$

We observe that the maximum value of $f(x, y)$ is 15 at the comer point $\mathrm{D}(0,5)$.

Q. 3 Maximize $z = 2x + 3y$; subject to the constraints:

$2x + y \leq 4$ ... (i)

$4x - y \leq 4$ ... (ii)

$x \geq 0, y \geq 0$ ... (iii)

Solution:

$$2x + y \leq 4 \text{ ... (i)}$$
$$4x - y \leq 4 \text{ ... (ii)}$$
$$x \geq 0, y \geq 0 \text{ ... (iii)}$$

The associated equation of inequality (i) is $2x + y = 4$ and it can be written in the form $\frac{x}{2} + \frac{y}{4} = 1$. This line intersects the x -axis and y-axis at point $A(2,0)$ and $B(0,4)$ respectively. Fig. 3(a)

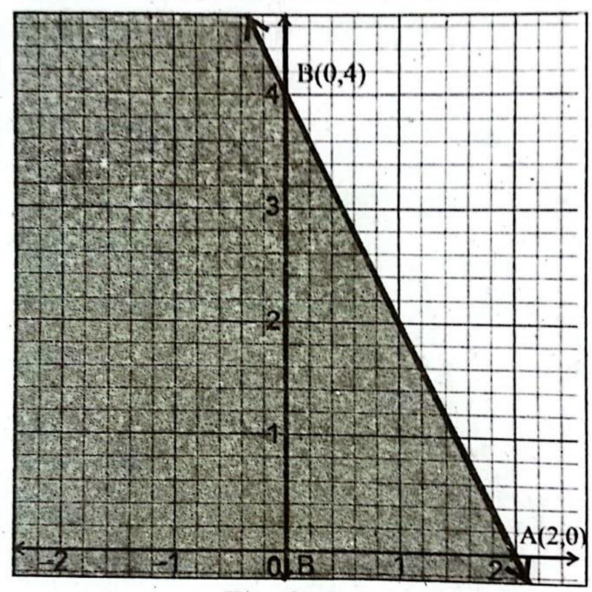

Fig. 3(a)

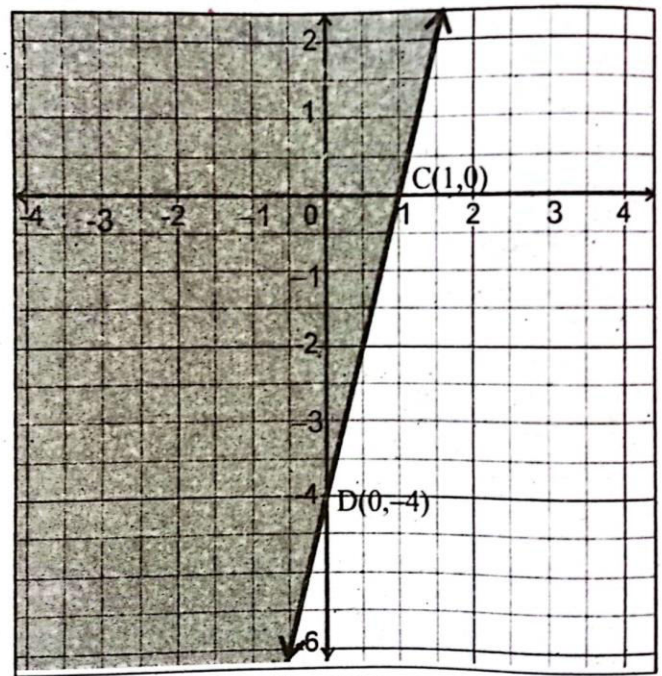

Fig. 3(b)

Now associated equation of in equality (ii) is $4x - y = 4$ and it can be written in the form $\frac{x}{1} + \frac{y}{-4} = 1$.This line intersects the x -axis and y-axis at point $C(1,0)$ and $D(0,-4)$ respectively. The sketch of inequality (ii) is shown as shaded region in the Fig.3(b). The graph of the inequalities in (iii) is the intersection region of graphs of $x \geq 0$ and $y \geq 0$ which is the first quadrant shown as shaded region in Fig.3(c).

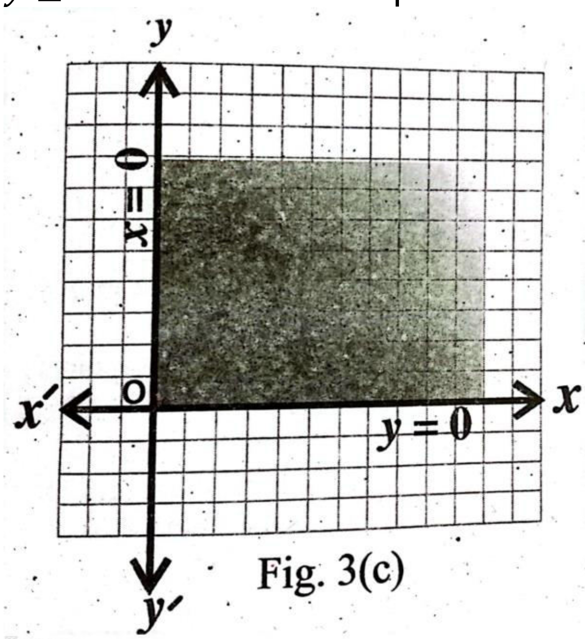

Fig. 3(c)

Now the solution region of the given system of inequalities is the intersection of the graphs indicated in the Fig. 3 (a) 3 (b) and 3(c) which is shown as shaded region in the Fig.3(d)
We observe that solution region is bounded region and its comer points are $\mathrm{B}(0,4)$, $\mathrm{A}(1,0), \mathrm{O}(0,0)$ and $\mathrm{E}\left(\frac{4}{3},\frac{4}{3}\right)$.
The point of intersection of corresponding lines of inequalities (i) and (ii) which can be obtained by solving the corresponding lines simultaneously.

$$2x + y = 4$$
$$4x - y = 4$$

Adding (iv) and (v) we get

$$2x + y + 4x - y = 4 + 4$$
$$6x = 8 \Rightarrow x = \frac{8}{6} = \frac{4}{3} = 1\frac{1}{3}$$

Put it in equation (Iv)

$$2\left(\frac{4}{3}\right) + y = 4 \Rightarrow y = 4 - \frac{8}{3} = \frac{12-8}{3} = \frac{4}{3} = 1\frac{1}{3}$$

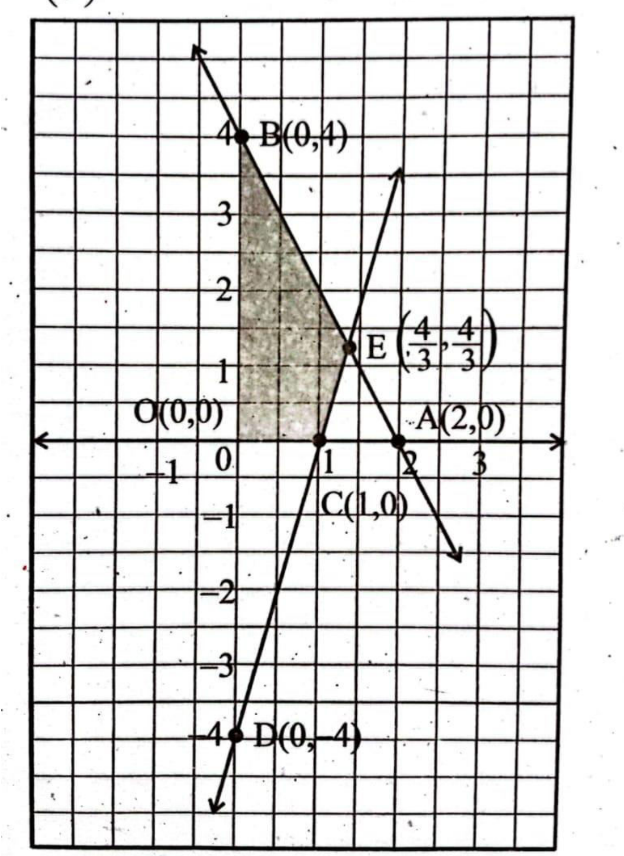

Fig. 3(d)

Fig. 3(d)
Now, we find the value of $f(x, y) = 2x + 3y$ at comer points.

Comer points	$f(x, y) = 2x + 3y$
$\mathrm{B}(0,4)$,	$f(0,4) = 2(0) + 3(4) = 0 + 12 = 12$
A(1,0)	$f(1.0) = 2(1) + 3(0) = 2 + 0 - 2$
O(0,0)	$f(0,0) = 2(0) + 3(0) = 0 + 0 = 0$
B $\left(\frac{4}{3}, \frac{4}{3}\right)$	$f\left(\frac{4}{3} \cdot \frac{4}{3}\right) = 2\left(\frac{4}{3}\right) + 3\left(\frac{4}{3}\right)$ $= \frac{8}{3} + 4 = \frac{8+12}{3} = \frac{20}{3} = 6.666$

We observe that the value of function is maximum 12 at comer point $\mathrm{B}(0,4)$.

Q. 4 Minimize $z = 2x + y$: subject to the constraints:

$$x + y \geq 3, 7x + 5y \leq 35, x \geq 0, y \geq 0$$

Solution:
The constraints are

$$x + y \geq 3$$
$$7x + 5y \leq 35$$
$$x \geq 0, y \geq 0$$

The sketch of inequality (i) is shown as shaded region in the figure 4(a) by joining the points $\mathrm{A}(3,0)$ and $\mathrm{B}(0,3)$ which are x and y intercepts respectively
The sketch of inequality (ii) is shown as shaded region in the figure 4(b) by joining the points $\mathrm{C}(5,0)$ and $\mathrm{D}(0,7)$ which are x and y intercepts respectively
The sketch of inequalities in (iii) is shown as shaded region in the figure 4(c)
Now the solution region of the given systems of inequalities is the intersection of the graphs indicated in the figure 4(a), 4(b), 4(c) and is shown as shaded region ACDB in the figure 4(d).

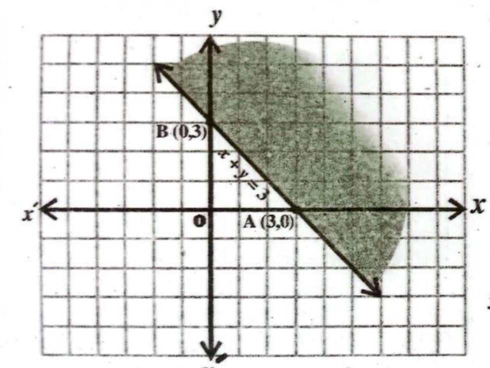

Fig. 4(a)

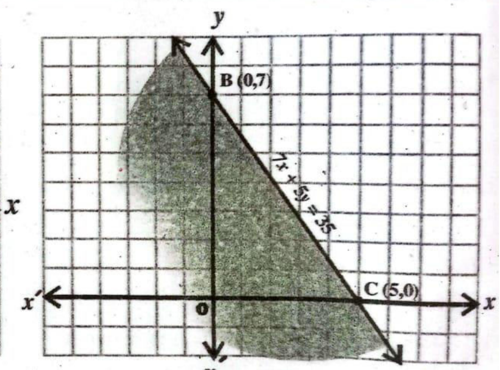

Fig. 4(b)

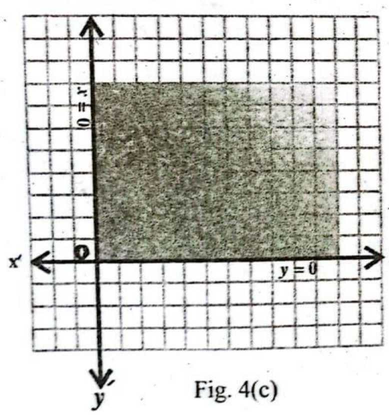

Fig. 4(c)

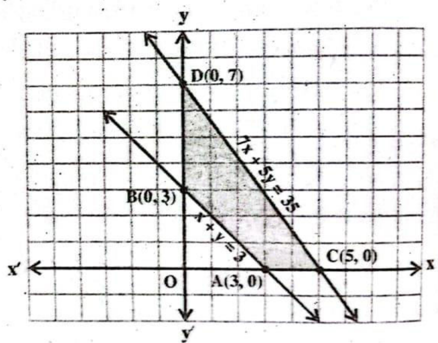

Fig. 4(d)

So the comer points are $\mathrm{A}, \mathrm{C}, \mathrm{D}$ and B .
Thus comer points are $\mathrm{A}(3,0), \mathrm{C}(0,5), \mathrm{D}(0,7)$ and $\mathrm{B}(0,3)$.
Now find the value of $f(x, y)$ at the comer's points, we have

Corner	Points	Ha.5) $2x =$
$\mathrm{A}(3,0)$		$/(3.0) = 2(3) + 0 = 6$
		$f(5.0) = 2(5) + 0 = 10$
D(0.7)		$f(0,7) = 2(0) + 7 = 7$
B(0.3)		$/(0.3) - 2(0) + 3 = 3$

We observe that the minimum value of $f(x, y)$ is 3 at the comer point $\mathrm{B}(0,3)$.
Q. 5 Maximize the function defined as: $f(x, y) = 2x + 3y$ subject to the constraints:

$$2x + y \leq 8; x + 2y \leq 14; x \geq 0; y \geq 0$$

Solution:
The constraints are

$$2x + y \leq 8$$
$$x + 2y \leq 14$$
$$x \geq 0, y \geq 0$$

The sketch of inequality (i) is shown as shaded region in the figure $S(a)$ by joining the points $A(4,0)$ and $B(0,8)$ which are x and y intereepts respectively The sketch of inequality (ii) is shown as shaded region in the figure 5 (b) by joining the points $\mathrm{C}(14,0)$ and $\mathrm{D}(0,7)$ which are x and y intercepts respectively The sketch of in equalities in (iii) is shown as shaded region in the figure $5(c)$
Now the solution region of the given systems of incqualities is the intersection of the graphs indicated in the figure 5(a). 5(b), 5(c) and is shown as shaded region OAED in the figure 5(d).

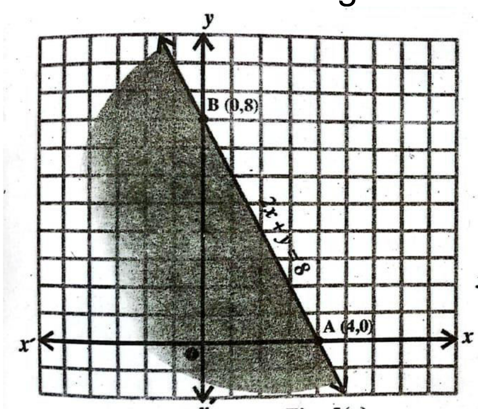

Fig. 5(a)

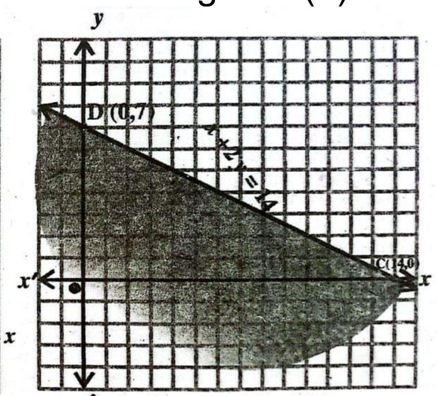

Fig. 5(b)

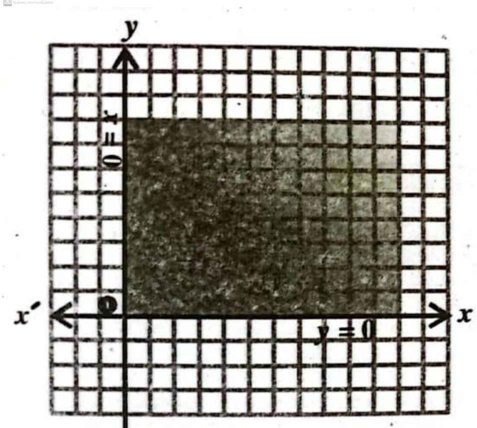

Fig. 5(c)

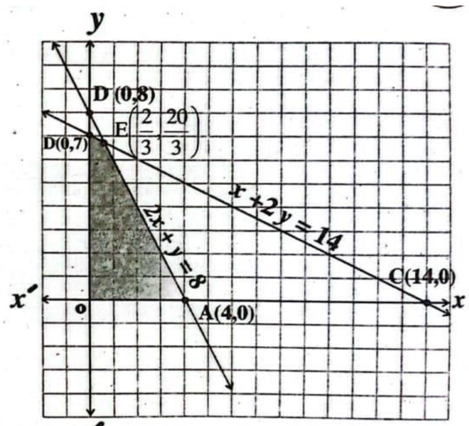

Fig. 5(d)

So the comer points are $\mathrm{O}, \mathrm{A}, \mathrm{E}$ and D .
For the comer point E, solving the line $2x + y = 8$ and $x + 2y = 14$ simultaneously, we have the interection point $\mathrm{E}\left(\frac{2}{3}, \frac{20}{3}\right)$

Thes comer points are $\mathrm{O}(0,0), \mathrm{A}(4,0), \mathrm{E}\left(\frac{2}{3}, \frac{20}{3}\right)$ and $\mathrm{D}(0,7)$.
Now find the value of fir y) at the corner points, we have

Corner point	
0(0.0)	$f(0,0)=2(0)+3(0)=0$
$\mathrm{A}(4,0)$	$f(4,0)=2(4)+3(0)=8$
E $\left(\frac{2}{3}, \frac{20}{3}\right)$	$\left(\frac{2}{3} \cdot \frac{20}{3}\right)=2\left(\frac{2}{3}\right)+3\left(\frac{20}{3}\right)=\frac{4}{3}+\frac{20}{1}$ $=\frac{4+60}{3}=\frac{64}{3}=21 \frac{1}{3}$
D(0,7)	$\mathrm{f}(0.7)=2(0)+3(7)=21$

$$D(0,7) \quad f(0,7)=2(0)+3(7)=21$$

We observe that the maximum value of $f(x, y)$ is " $21 \frac{1}{3}$ " at the comer point $E\left(\frac{2}{3}, \frac{20}{3}\right.$

$$D(0,7) \quad f(0,7)=2(0)+3(7)=21$$

We observe that the maximum value of $f(x, y)$ is " $21 \frac{1}{3}$ " at the comer point $E\left(\frac{2}{3}, \frac{20}{3}\right.$

Q.6 Minimize $z=3x+y$; subject to the constraints:

$$\begin{aligned}
& 3x+5y \geq 15 \\
& x+6y \geq 9 \\
& x \geq 0, ; y \geq 0
\end{aligned}$$

Solution:
The constraints are

$$\begin{gathered}
3x+5y \geq 15 \ldots \ldots \text{ (i)} \\
x+6y \geq 9. \\
x \geq 0, y \geq 0
\end{gathered}$$

The associated equation of inequality (i) is $3x+5y=15$ and it can be written in the form $\frac{x}{5}+\frac{y}{3}=1$. This line intersects the x -axis and y-axis at point $A(5,0)$ and $B(0,3)$

respectively. The sketch of inequality (i) is shown as shaded region in the Fig.6(a).

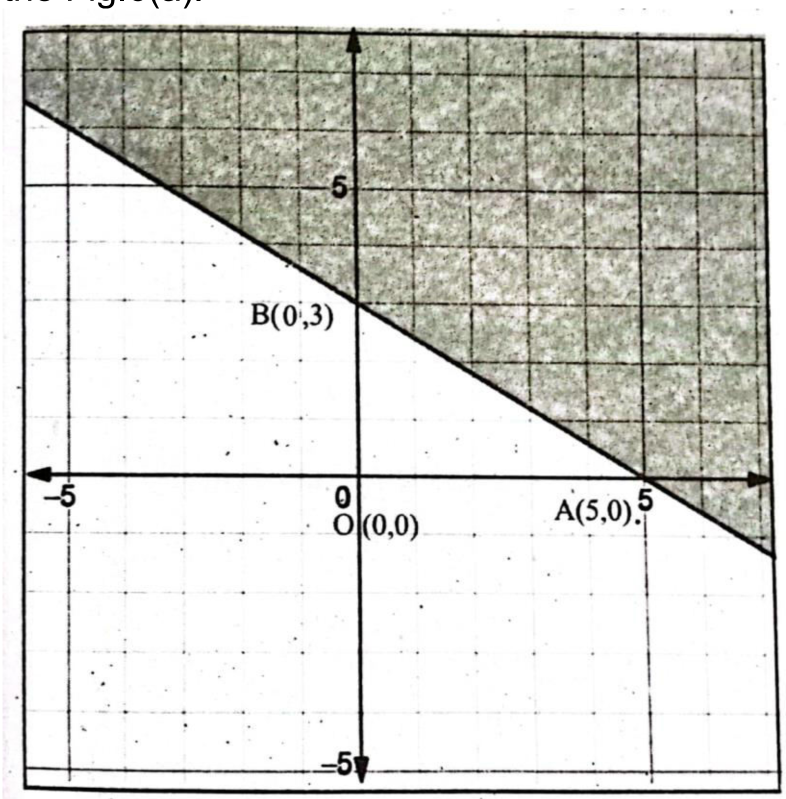

Fig. 6(a)

Fig. 6(a)

The associated equation of inequality (ii) is $x + 6y = 9$ and it can be written in the form $\frac{x}{9} + \frac{y}{3/2} = 1$. This line intersects the x -axis and y-axis at point $C(9,0)$ and $D(0,3/2)$ respectively. The sketch of inequality (ii) is shown as shaded region in the Fig.6(b).

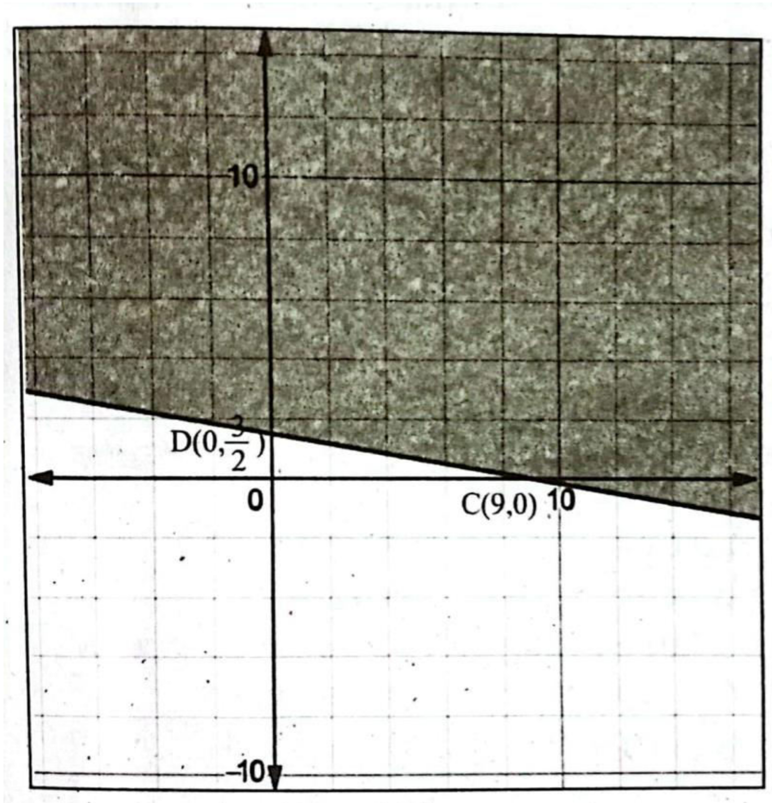

Fig. 6(b)

The graph of the inequalities in (ii) is the intersection region of graphs of $x \geq 0$ and $y \geq 0$ which is the first quadrant shown as shaded region in Fig.6(c).

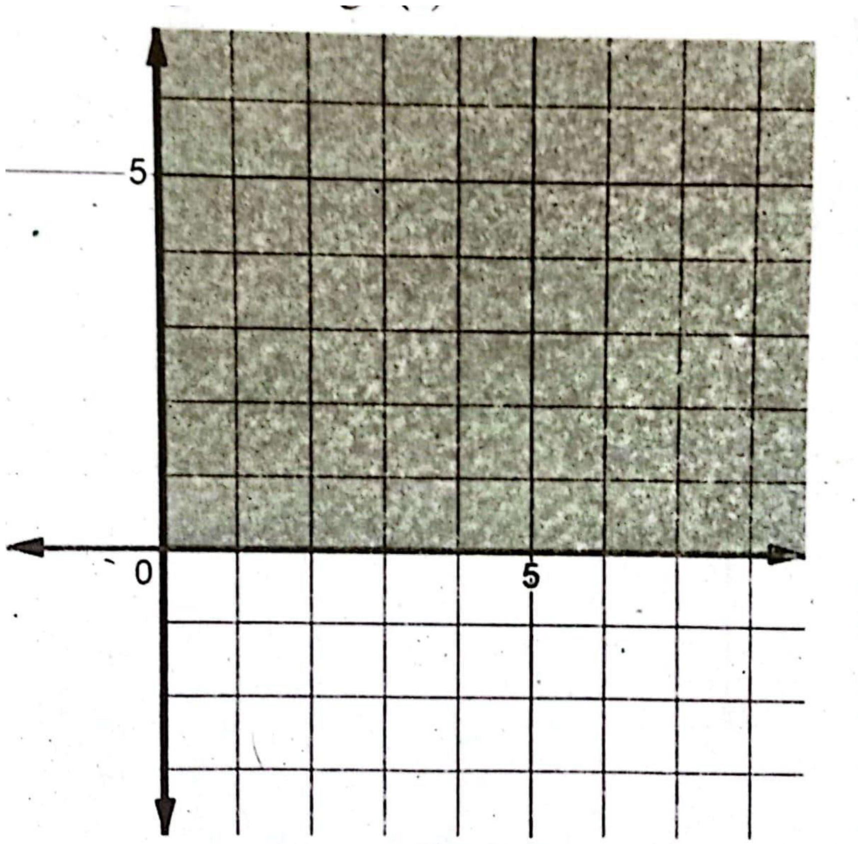

Fig. 6(c)

Fig. 6(c)

Now the solution region of the given system of inequalities is the intersection of the graphs indicated in the Fig.6(a) ,6(b) and 6(c) which is shown as shaded region in the Fig.6(d)

We observe that solution region is unbounded and it comer points are $B(0,3)$ $C(9,0)$ and $E\left(\frac{45}{13}, \frac{12}{13}\right)$. Point E is the point of intersection of corresponding lines of inequalities (i) and (ii) which can be obtained by solving the corresponding equations simultaneously.

$$3x + 5y = 15 \ldots \ldots (\text{ iii) })$$

$$x + 6y = 9 \ldots \ldots \ldots (\text{ iv })$$

Subtract eq.(iii) from 3 times the eq.(iv)

$$3(x + 6y) - (3x + 5y) = 3(9) - 15$$

$$3x + 18y - 3x - 5y = 27 - 15$$

$$13y = 12 \Rightarrow y = \frac{12}{13}$$

Put it in Eq.(iv)

$$x + 6\left(\frac{12}{13}\right) = 9 \Rightarrow x = 9 - \frac{72}{13} \Rightarrow x = \frac{117 - 72}{13} = \frac{45}{13}$$

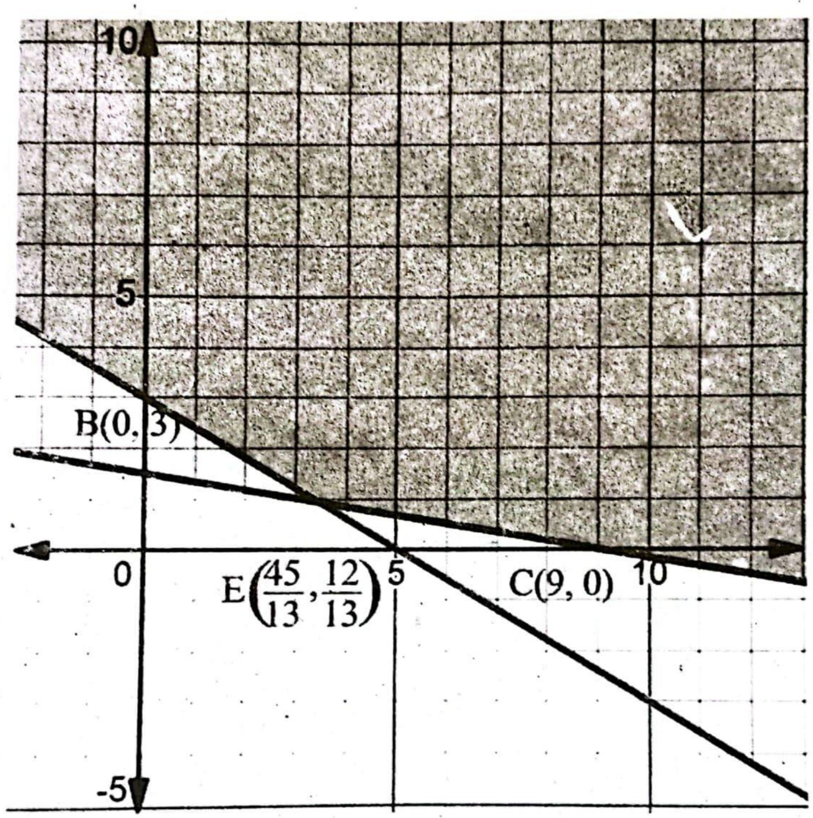

Fig. 6(d)

Fig. 6(d)

We find the value of $(x, y) = 3x + y$ at comer points.

Comer points	$f(x, y) = 3x + y$
$B(0,3)$	$f(0,3) = 3(0) + 3 = 0 + 3 = 3$
$C(9,0)$	$f(9,0) = 3(9) + 0 = 27 + 0 = 27$
$E\left(\frac{45}{13}, \frac{12}{13}\right)$	$f\left(\frac{45}{13}, \frac{12}{13}\right) =$
	$= 3\left(\frac{45}{13}\right) + \frac{12}{13} = \frac{135}{13} + \frac{12}{13}$ $= \frac{147}{13} = 11.3$

We observe that the value of function is minimum 3 at comer point $B(0,3)$ while the value of function is maximum 27 at comer point $C(9,0)$