EXERCISE 4.1

1. If $f(x) = 2x + 5$, $g(x) = 3x - 2$, then find
   (i) $f(x) + g(x)$ (ii) $f(x) - g(x)$
2. If $f(x) = x + 2$, $g(x) = 2x + 4$, then find
   (i) $f(x) \cdot g(x)$ (ii) $g(x) \cdot f(x)$ (iii) $\dfrac{f(x)}{g(x)}$ (iv) $\dfrac{g(x)}{f(x)}$
3. For the functions f and g, find
   (a) $(fog)(x)$ (b) $(gof)(x)$ (c) $(fof)(x)$ (d) $(gog)(x)$

   Where,

   (i) $f(x) = 2x + 3$; $g(x) = x^3$ (ii) $f(x) = \dfrac{2}{x}, x \neq 0$; $g(x) = 2x^2 - 1$

   (iii) $f(x) = 2x - 1$; $g(x) = \dfrac{x+1}{2}$
4. Find the value of k, such that $(fog)(x) = (gof)(x)$, where $f(x) = 3x + 2$, $g(x) = 6x - k$.
5. Given that $f(x) = 3x + 2$ and $g(x) = 2x + 3$. Find
   (i) $g(f(4))$ (ii) $f(f(3))$ (iii) $f(g(-2))$
6. Find $f^{-1}(x)$ in each of the following:
   (i) $f(x) = 2x - 3$ (ii) $f(x) = 4x^3 - 1$

   (iii) $f(x) = \sqrt{x-1}, x \geq 1$ (iv) $f(x) = \dfrac{x+1}{3x-2}, x \neq \dfrac{2}{3}$
7. The functions f and g are defined such that $f(x) = 4x + 2$ and $g(x) = 6x - 18$.
   (i) Find $f^{-1}(x)$ and $g^{-1}(x)$. (ii) Find x if $f^{-1}(x) = g^{-1}(x)$.
8. Verify that $f(f^{-1}(x)) = f^{-1}(f(x)) = x$
   (i) $f(x) = x - 6$ (ii) $f(x) = 7x - 4$

   (iii) $f(x) = \dfrac{x-3}{4}$ (iv) $f(x) = \dfrac{x-4}{x+2}, x \neq -2$
9. Without finding $f^{-1}(x)$, find domain and range of $f^{-1}(x)$:
   (i) $f(x) = 12x - 3$ (ii) $f(x) = \dfrac{1}{2}x + 8$

   (iii) $f(x) = \dfrac{x}{1+x}, x \neq -1$ (iv) $f(x) = \sqrt{x-2}, x \geq 2$
10. Given that $f(x) = x^2 + 9$ and $g(x) = x + 21$. Find the values of a such that: $f(a) = g(a)$

EXERCISE 4.1 - FUNCTIONS AND COMPOSITIONS

Complete Step-by-Step Solutions

Question 1: Addition and Subtraction of Functions

Given: $\mathbf{f(x) = 2x + 5,\ g(x) = 3x - 2}$

(i) $\mathbf{f(x) + g(x)}$

Step 1: Add the functions

$$f(x) + g(x) = (2x + 5) + (3x - 2)$$

Step 2: Combine like terms

$$= 5x + 3$$

Answer: $5x + 3$

(ii) $\mathbf{f(x) - g(x)}$

Step 1: Subtract the functions

$$f(x) - g(x) = (2x + 5) - (3x - 2)$$

Step 2: Combine like terms

$$= 2x + 5 - 3x + 2 = -x + 7$$

Answer: $-x + 7$

Question 2: Multiplication and Division of Functions

Given: $\mathbf{f(x) = x + 2,\ g(x) = 2x + 4}$

(i) $\mathbf{f(x)g(x)}$

Step 1: Multiply the functions

$$f(x)g(x) = (x + 2)(2x + 4)$$

Step 2: Expand

$$= 2x^2 + 4x + 4x + 8 = 2x^2 + 8x + 8$$

Answer: $2x^2 + 8x + 8$

(ii) $\mathbf{g(x)f(x)}$

Step 1: Multiplication is commutative

$$g(x)f(x) = f(x)g(x)$$

Answer: $2x^2 + 8x + 8$

(iii) $\mathbf{\frac{f(x)}{g(x)}}$

Step 1: Divide the functions

$$\frac{f(x)}{g(x)} = \frac{x + 2}{2x + 4}$$

Step 2: Factor denominator

$$= \frac{x + 2}{2(x + 2)}$$

Step 3: Cancel $(x + 2)$, $x \neq -2$

$$= \frac{1}{2}$$

Answer: $\frac{1}{2}$, $x \neq -2$

(iv) $\frac{\mathbf{g(x)}}{\mathbf{f(x)}}$

Step 1: Divide the functions

$$\frac{g(x)}{f(x)} = \frac{2x+4}{x+2}$$

Step 2: Factor numerator

$$= \frac{2(x+2)}{x+2}$$

Step 3: Cancel $(x+2)$, $x \neq -2$

$$= 2$$

Answer: 2, $x \neq -2$

Question 3: Composition of Functions

(i) $\mathbf{f(x) = 2x + 3,\ g(x) = x^3}$

(a) $(\mathbf{f} \circ \mathbf{g})(\mathbf{x}) = \mathbf{f}(\mathbf{g}(\mathbf{x}))$

Step 1: Substitute $g(x)$ into $f(x)$

$$f(g(x)) = f(x^3) = 2(x^3) + 3$$

Answer: $2x^3 + 3$

(b) $(\mathbf{g} \circ \mathbf{f})(\mathbf{x}) = \mathbf{g}(\mathbf{f}(\mathbf{x}))$

Step 1: Substitute $f(x)$ into $g(x)$

$$g(f(x)) = g(2x+3) = (2x+3)^3$$

Answer: $(2x+3)^3$

(c) $(\mathbf{f} \circ \mathbf{f})(\mathbf{x}) = \mathbf{f}(\mathbf{f}(\mathbf{x}))$

Step 1: Substitute $f(x)$ into itself

$$f(f(x)) = f(2x+3) = 2(2x+3) + 3 = 4x + 6 + 3 = 4x + 9$$

Answer: $4x + 9$

(d) $(\mathbf{g} \circ \mathbf{g})(\mathbf{x}) = \mathbf{g}(\mathbf{g}(\mathbf{x}))$

Step 1: Substitute $g(x)$ into itself

$$g(g(x)) = g(x^3) = (x^3)^3 = x^9$$

Answer: x^9

(ii) $\mathbf{f(x) = \frac{2}{x},\ g(x) = 2x^2 - 1}$

(a) $(\mathbf{f} \circ \mathbf{g})(\mathbf{x})$

Step 1: $f(g(x)) = f(2x^2 - 1) = \frac{2}{2x^2-1}$

Answer: $\frac{2}{2x^2-1}$, $2x^2 - 1 \neq 0$

(b) $(\mathbf{g} \circ \mathbf{f})(\mathbf{x})$

Step 1: $g(f(x)) = g\left(\frac{2}{x}\right) = 2\left(\frac{2}{x}\right)^2 - 1$

$$= \frac{8}{x^2} - 1$$

Answer: $\frac{8}{x^2} - 1$, $x \neq 0$

(c) $(\mathbf{f} \circ \mathbf{f})(\mathbf{x})$

Step 1: $f(f(x)) = f\left(\frac{2}{x}\right) = \frac{2}{2/x} = 2 \cdot \frac{x}{2} = x$

Answer: x, $x \neq 0$

(d) $(\mathbf{g} \circ \mathbf{g})(\mathbf{x})$

Step 1: $g(g(x)) = g(2x^2 - 1) = 2(2x^2 - 1)^2 - 1$

$$= 2(4x^4 - 4x^2 + 1) - 1 = 8x^4 - 8x^2 + 2 - 1$$

Answer: $8x^4 - 8x^2 + 1$

(iii) $\mathbf{f(x) = 2x - 1,\ g(x) = \frac{x+1}{2}}$

(a) $(\mathbf{f} \circ \mathbf{g})(\mathbf{x})$

Step 1: $f(g(x)) = f\left(\frac{x+1}{2}\right) = 2\left(\frac{x+1}{2}\right) - 1$

$$= x + 1 - 1 = x$$

Answer: x

(b) $(\mathbf{g} \circ \mathbf{f})(\mathbf{x})$

Step 1: $g(f(x)) = g(2x - 1) = \frac{(2x-1)+1}{2} = \frac{2x}{2} = x$

Answer: x

(c) $(\mathbf{f} \circ \mathbf{f})(\mathbf{x})$

Step 1: $f(f(x)) = f(2x - 1) = 2(2x - 1) - 1$

$$= 4x - 2 - 1 = 4x - 3$$

Answer: $4x - 3$

(d) $(\mathbf{g} \circ \mathbf{g})(\mathbf{x})$

Step 1: $g(g(x)) = g\left(\frac{x+1}{2}\right) = \frac{\frac{x+1}{2}+1}{2}$

$$= \frac{\frac{x+1+2}{2}}{2} = \frac{x+3}{4}$$

Answer: $\frac{x+3}{4}$

Question 4: Find k such that $(\mathbf{f} \circ \mathbf{g})(\mathbf{x}) = (\mathbf{g} \circ \mathbf{f})(\mathbf{x})$

Given: $\mathbf{f(x) = 3x + 2,\ g(x) = 6x - k}$

Step 1: Find $f(g(x))$

$$f(g(x)) = f(6x - k) = 3(6x - k) + 2 = 18x - 3k + 2$$

Step 2: Find $g(f(x))$

$$g(f(x)) = g(3x + 2) = 6(3x + 2) - k = 18x + 12 - k$$

Step 3: Set equal

$$18x - 3k + 2 = 18x + 12 - k – 3k + 2 = 12 - k – 2k = 10k = -5$$

Answer: $k = -5$

Question 5: Function Evaluation

Given: $\mathbf{f(x) = 3x + 2,\ g(x) = 2x + 3}$

(i) $\mathbf{g(f(4))}$

Step 1: Find $f(4)$

$$f(4) = 3(4) + 2 = 14$$

Step 2: Find $g(14)$

$$g(14) = 2(14) + 3 = 31$$

Answer: 31

(ii) $\mathbf{f(f(3))}$

Step 1: Find $f(3)$

$$f(3) = 3(3) + 2 = 11$$

Step 2: Find $f(11)$

$$f(11) = 3(11) + 2 = 35$$

Answer: 35

(iii) $\mathbf{f(g(-2))}$

Step 1: Find $g(-2)$

$$g(-2) = 2(-2) + 3 = -1$$

Step 2: Find $f(-1)$

$$f(-1) = 3(-1) + 2 = -1$$

Answer: -1

Question 6: Find Inverse Functions

(i) $\mathbf{f(x) = 2x - 3}$

Step 1: Replace $f(x)$ with y

$$y = 2x - 3$$

Step 2: Solve for x

$$y + 3 = 2xx = \frac{y+3}{2}$$

Step 3: Replace y with x

$$f^{-1}(x) = \frac{x+3}{2}$$

Answer: $f^{-1}(x) = \frac{x+3}{2}$

(ii) $\mathbf{f(x) = 4x^3 - 1}$

Step 1: $y = 4x^3 - 1$

Step 2: Solve for x

$$y + 1 = 4x^3x^3 = \frac{y+1}{4}x = \sqrt[3]{\frac{y+1}{4}}$$

Answer: $f^{-1}(x) = \sqrt[3]{\frac{x+1}{4}}$

(iii) $\mathbf{f(x) = \sqrt{x-1}, x \geq 1}$

Step 1: $y = \sqrt{x-1}$

Step 2: Solve for x

$$y^2 = x - 1x = y^2 + 1$$

Step 3: Domain of f^{-1} = Range of $f = [0, \infty)$

Answer: $f^{-1}(x) = x^2 + 1, x \geq 0$

(iv) $\mathbf{f(x) = \frac{x+1}{3x-2}, x \neq \frac{2}{3}}$

Step 1: $y = \frac{x+1}{3x-2}$

Step 2: Solve for x

$$y(3x-2) = x + 13xy - 2y = x + 13xy - x = 2y + 1x(3y-1) = 2y + 1x = \frac{2y+1}{3y-1}$$

Answer: $f^{-1}(x) = \frac{2x+1}{3x-1}, x \neq \frac{1}{3}$

Question 7: Inverse Functions

Given: $\mathbf{f(x) = 4x + 2,\ g(x) = 6x - 18}$

(i) Find $\mathbf{f^{-1}(x)}$ **and** $\mathbf{g^{-1}(x)}$

For $f(x) = 4x + 2$**:**

$$y = 4x + 2y - 2 = 4xx = \frac{y-2}{4}f^{-1}(x) = \frac{x-2}{4}$$

For $g(x) = 6x - 18$**:**

$$y = 6x - 18y + 18 = 6xx = \frac{y+18}{6}g^{-1}(x) = \frac{x+18}{6}$$

(ii) Find x if $\mathbf{f^{-1}(x) = g^{-1}(x)}$

Step 1: Set equal

$$\frac{x-2}{4} = \frac{x+18}{6}$$

Step 2: Cross multiply

$$6(x-2) = 4(x+18)6x - 12 = 4x + 722x = 84x = 42$$

Answer: $x = 42$

Question 8: Verify $\mathbf{f(f^{-1}(x)) = f^{-1}(f(x)) = x}$

(i) $\mathbf{f(x) = x - 6}$

Step 1: Find $f^{-1}(x)$

$$y = x - 6 \rightarrow x = y + 6 \rightarrow f^{-1}(x) = x + 6$$

Step 2: $f(f^{-1}(x)) = f(x+6) = (x+6) - 6 = x$

Step 3: $f^{-1}(f(x)) = f^{-1}(x-6) = (x-6) + 6 = x$

Verified

(ii) $\mathbf{f(x) = 7x - 4}$

Step 1: $f^{-1}(x) = \frac{x+4}{7}$

Step 2: $f(f^{-1}(x)) = f\left(\frac{x+4}{7}\right) = 7\left(\frac{x+4}{7}\right) - 4 = x + 4 - 4 = x$

Step 3: $f^{-1}(f(x)) = f^{-1}(7x-4) = \frac{7x-4+4}{7} = \frac{7x}{7} = x$

Verified

(iii) $\mathbf{f(x) = \frac{x-3}{4}}$

Step 1: $f^{-1}(x) = 4x + 3$

Step 2: $f(f^{-1}(x)) = f(4x+3) = \frac{4x+3-3}{4} = x$

Step 3: $f^{-1}(f(x)) = f^{-1}\left(\frac{x-3}{4}\right) = 4\left(\frac{x-3}{4}\right) + 3 = x - 3 + 3 = x$

Verified

(iv) $\mathbf{f(x) = \frac{x-4}{x+2}, x \neq -2}$

Step 1: Find $f^{-1}(x)$

$$y = \frac{x-4}{x+2}y(x+2) = x - 4yx + 2y = x - 4yx - x = -4 - 2yx(y-1) = -4 - 2yx = \frac{-4-2y}{y-1} = \frac{2y+4}{1-y}f^{-1}(x) = \frac{2x+4}{1-x}$$

Step 2: $f(f^{-1}(x)) = f\left(\frac{2x+4}{1-x}\right)$
$= \frac{\frac{2x+4}{1-x} - 4}{\frac{2x+4}{1-x} + 2} = \frac{\frac{2x+4-4+4x}{1-x}}{\frac{2x+4+2-2x}{1-x}} = \frac{6x}{6} = x$

Verified

Question 9: Domain and Range of Inverse

Without finding inverse:

Property: Domain of f^{-1} = Range of f
Range of f^{-1} = Domain of f

(i) $\mathbf{f(x) = 12x - 3}$

Step 1: Domain of $f = (-\infty, \infty)$
Step 2: Range of $f = (-\infty, \infty)$

Answer: Domain of f^{-1}: $(-\infty, \infty)$, Range of f^{-1}: $(-\infty, \infty)$

(ii) $\mathbf{f(x) = \frac{1}{2}x + 8}$

Step 1: Domain of $f = (-\infty, \infty)$
Step 2: Range of $f = (-\infty, \infty)$

Answer: Domain of f^{-1}: $(-\infty, \infty)$, Range of f^{-1}: $(-\infty, \infty)$

Summary Table

1(i)	$5x + 3$
1(ii)	$-x + 7$
2(i)	$2x^2 + 8x + 8$
2(ii)	$2x^2 + 8x + 8$

2(iii)	$\frac{1}{2}$
2(iv)	2
3(i)(a)	$2x^3 + 3$
3(i)(b)	$(2x+3)^3$
3(i)(c)	$4x + 9$
3(i)(d)	x^9
3(ii)(a)	$\frac{2}{2x^2-1}$
3(ii)(b)	$\frac{8}{x^2} - 1$
3(ii)(c)	x
3(ii)(d)	$8x^4 - 8x^2 + 1$
3(iii)(a)	x
3(iii)(b)	x
3(iii)(c)	$4x - 3$
3(iii)(d)	$\frac{x+3}{4}$
4	$k = -5$
5(i)	31
5(ii)	35
5(iii)	-1
6(i)	$\frac{x+3}{2}$
6(ii)	$\sqrt[3]{\frac{x+1}{4}}$
6(iii)	$x^2 + 1$
6(iv)	$\frac{2x+1}{3x-1}$
7(i)	$\frac{x-2}{4}, \frac{x+18}{6}$
7(ii)	$x = 42$
8(i)-(iv)	All Verified
9(i)	Domain: $\mathbb{R}$, Range: $\mathbb{R}$
9(ii)	Domain: $\mathbb{R}$, Range: $\mathbb{R}$

EXERCISE 4.1 - FUNCTIONS AND COMPOSITIONS (Continued)

Question 9 (Continued): Domain and Range of Inverse

(iii) $\mathbf{f(x) = \frac{x}{1+x}, x \neq -1}$

Step 1: Domain of f: $x \neq -1 \to (-\infty, -1) \cup (-1, \infty)$

Step 2: Find Range of f

$$y = \frac{x}{1+x} y(1+x) = xy + xy = xy = x - xy = x(1-y)x = \frac{y}{1-y}$$

Step 3: For x to be defined, $y \neq 1$

- Range of f: $y \neq 1 \to (-\infty, 1) \cup (1, \infty)$

Step 4: Domain of f^{-1} = Range of f = $(-\infty, 1) \cup (1, \infty)$
Step 5: Range of f^{-1} = Domain of f = $(-\infty, -1) \cup (-1, \infty)$

Answer: Domain of f^{-1}: $(-\infty, 1) \cup (1, \infty)$
Range of f^{-1}: $(-\infty, -1) \cup (-1, \infty)$

(iv) $\mathbf{f(x) = \sqrt{x-2}, x \geq 2}$

Step 1: Domain of f: $[2, \infty)$

Step 2: Range of f: Since $x \geq 2$, $x - 2 \geq 0$, so $f(x) \geq 0$

- Range: $[0, \infty)$

Step 3: Domain of f^{-1} = Range of f = $[0, \infty)$
Step 4: Range of f^{-1} = Domain of f = $[2, \infty)$

Answer: Domain of f^{-1}: $[0, \infty)$
Range of f^{-1}: $[2, \infty)$

Question 10: Find values of a such that $\mathbf{f(a) = g(a)}$

Given: $\mathbf{f(x) = x^2 + 9,\ g(x) = x + 21}$

Step 1: Set $f(a) = g(a)$

$$a^2 + 9 = a + 21$$

Step 2: Rearrange

$$a^2 - a + 9 - 21 = 0a^2 - a - 12 = 0$$

Step 3: Factor

$$(a - 4)(a + 3) = 0$$

Step 4: Solve

$$a = 4 \text{or} a = -3$$

Answer: $a = 4$ or $a = -3$

Summary Table (Continued)

9(iii)	Domain: $(-\infty, 1) \cup (1, \infty)$, Range: $(-\infty, -1) \cup (-1, \infty)$
9(iv)	Domain: $[0, \infty)$, Range: $[2, \infty)$
10	$a = 4$ or $a = -3$

Key Concepts Summary

1. Domain and Range of Inverse Function

- **Domain of f^{-1} = Range of f**
- **Range of f^{-1} = Domain of f**

2. Composition of Functions

- $(f \circ g)(x) = f(g(x))$
- Domain of composition: x in domain of g and $g(x)$ in domain of f

3. Inverse Function Properties

- $f(f^{-1}(x)) = f^{-1}(f(x)) = x$
- A function must be one-to-one to have an inverse