EXERCISE 1.2

Q.1

Find the value of x and y in each of the following:

(i) $\mathbf{x + iy + 2 - 3i = i(5 - i)(3 + 4i)}$

Solution:

$$x + iy + 2 - 3i = i(5 - i)(3 + 4i) \Rightarrow x + 2 + iy - 3i = i(15 + 20i - 3i - 4i^2) \Rightarrow (x + 2) + i(y - 3) = i\{15 + 17i - 4(-1)\} \Rightarrow (x$$

Comparing real and imaginary parts:

$$x + 2 = -17 \text{(i)} y - 3 = 19 \text{(ii)}$$

From (i):

$$x = -19$$

From (ii):

$$y = 22$$

Answer: $x = -19$, $y = 22$

(ii) $\mathbf{(x + iy)(1 - i) = (2 - 3i)(-5 + 5i)\left(-\frac{3i}{5}\right)}$

Solution:

$$(x + iy)(1 - i) = (2 - 3i)(-5 + 5i)\left(-\frac{3i}{5}\right) \Rightarrow x - ix + iy - i^2 y = (-10 + 10i + 15i - 15i^2)\left(-\frac{3i}{5}\right) \Rightarrow x - ix + iy - (-1)y = ($$

Comparing real and imaginary parts:

$$x + y = 15 \text{(i)} y - x = -3 \text{(ii)}$$

Adding (i) and (ii):

$$2y = 12 \Rightarrow y = 6$$

Putting $y = 6$ in (i):

$$x + 6 = 15 \Rightarrow x = 9$$

Answer: $x = 9$, $y = 6$

(iii) $\mathbf{\frac{x}{2+i} + \frac{y}{3-i} = 4 + 5i}$

Solution:

$$\frac{x}{2+i}+\frac{y}{3-i}=4+5i\Rightarrow \frac{x}{2+i}\cdot\frac{2-i}{2-i}+\frac{y}{3-i}\cdot\frac{3+i}{3+i}=4+5i\Rightarrow \frac{x(2-i)}{2^2-i^2}+\frac{y(3+i)}{3^2-i^2}=4+5i\Rightarrow \frac{2x-ix}{5}+\frac{3y+iy}{10}=4+5i\Rightarrow$$

Comparing real and imaginary parts:

$$4x+3y=40(\text{i})y-2x=50(\text{ii})$$

From (ii):

$$y=50+2x(\text{iii})$$

Using (iii) in (i):

$$4x+3(50+2x)=404x+150+6x=4010x=-110x=-11$$

Putting $x=-11$ in (iii):

$$y=50+2(-11)=28$$

Answer: $x=-11$, $y=28$

Q.2

If $z_1=-13+24i$ and $z_2=x+iy$, find the values of x and y such that

$$z_1-z_2=-27+15i.$$

Solution:

$$z_1-z_2=-27+15i\Rightarrow (-13+24i)-(x+iy)=-27+15i\Rightarrow -13+24i-x-iy=-27+15i\Rightarrow (-13-x)+i(24-y)=-27$$

Comparing real and imaginary parts:

$$-13-x=-27(\text{i})24-y=15(\text{ii})$$

From (i):

$$x=14$$

From (ii):

$$y=9$$

Answer: $x=14$, $y=9$

Q.3

Find the real values of x and y if:

(i) $\left(\mathbf{x+iy}\right)^{\mathbf{2}}=\mathbf{25+60i}$

Solution:

$$(x+iy)^2=25+60i \Rightarrow x+iy=\sqrt{25+60i}$$

Let $z=a+ib=25+60i$. Then:

$$|z|=\sqrt{25^2+60^2}=\sqrt{625+3600}=\sqrt{4225}=65$$

Using the square root formula:

$$\sqrt{z}=\pm\left(\sqrt{\frac{|z|+a}{2}}+i\frac{b}{|b|}\sqrt{\frac{|z|-a}{2}}\right)\sqrt{25+60i}=\pm\left(\sqrt{\frac{65+25}{2}}+i\sqrt{\frac{65-25}{2}}\right)=\pm\left(\sqrt{45}+i\sqrt{20}\right)=\pm\left(3\sqrt{5}+i2\sqrt{5}\right)$$

Thus:

$$x+iy=\pm 3\sqrt{5}\pm i2\sqrt{5}$$

Answer: $x=\pm 3\sqrt{5}$, $y=\pm 2\sqrt{5}$

(ii) $\left(\mathbf{x+iy}\right)^{\mathbf{2}}=\mathbf{64+48i}$

Solution:

$$(x+iy)^2=64+48i \Rightarrow x+iy=\sqrt{64+48i}$$

Let $z=a+ib=64+48i$. Then:

$$|z|=\sqrt{64^2+48^2}=\sqrt{4096+2304}=\sqrt{6400}=80$$

Using the square root formula:

$$\sqrt{64+48i}=\pm\left(\sqrt{\frac{80+64}{2}}+i\sqrt{\frac{80-64}{2}}\right)=\pm\left(\sqrt{72}+i\sqrt{8}\right)=\pm\left(6\sqrt{2}+i2\sqrt{2}\right)$$

Thus:

$$x+iy=\pm 6\sqrt{2}\pm i2\sqrt{2}$$

Answer: $x=\pm 6\sqrt{2}$, $y=\pm 2\sqrt{2}$

(iii) $\left(\mathbf{x+iy}\right)^{\mathbf{2}}=\frac{\mathbf{2i-3}}{\mathbf{3+i}}$

Solution:

$$\left(x+iy\right)^2=\frac{2i-3}{3+i}$$

Rationalize:

$$= \frac{2i-3}{3+i} \times \frac{3-i}{3-i} = \frac{6i-2i^2-9+3i}{3^2-i^2} = \frac{9i+2-9}{9+1} = \frac{-7+9i}{10} = -\frac{7}{10} + \frac{9}{10}i$$

Thus:

$$x + iy = \sqrt{-\frac{7}{10} + \frac{9}{10}i}$$

Let $z = a + ib = -0.7 + 0.9i$. Then:

$$\mid z \mid = \sqrt{(-0.7)^2 + (0.9)^2} = \sqrt{0.49 + 0.81} = \sqrt{1.3} \approx 1.1402$$

Using the square root formula:

$$\sqrt{z} = \pm \left(\sqrt{\frac{1.1402 - 0.7}{2}} + i\sqrt{\frac{1.1402 + 0.7}{2}} \right) = \pm \left(\sqrt{0.2201} + i\sqrt{0.9201} \right) \approx \pm(0.4691 + i0.9592)$$

Answer: $x \approx \pm 0.4691$, $y \approx \pm 0.9592$

Q.4

If $z_1 = 2 + 3i$ and $z_2 = 1 - \alpha$, find the value of α such that

$$Im(z_1 z_2) = 7.$$

Solution:

$$z_1 = 2 + 3i, z_2 = 1 - \alpha Im[(2+3i)(1-\alpha)] = 7 Im[2 - 2\alpha + 3i - 3\alpha i] = 7 Im[(2-2\alpha) + i(3-3\alpha)] = 73 - 3\alpha = 7 - 3\alpha = 4\alpha =$$

Answer: $\alpha = -\frac{4}{3}$

Q.5

If $z_1 = x + iy$ and $z_2 = a + ib$, find x, y, a and b such that

$$z_1 + z_2 = 10 + 4i$$

and

$$z_1 - z_2 = 6 + 2i.$$

Solution:

Adding the two equations:

$$2z_1 = 16 + 6i z_1 = 8 + 3i \Rightarrow x = 8, y = 3$$

Using $z_1 + z_2 = 10 + 4i$:

$$(8 + 3i) + z_2 = 10 + 4i z_2 = 2 + i \Rightarrow a = 2, b = 1$$

Answer: $x = 8,\ y = 3,\ a = 2,\ b = 1$

Q.6

Show that $\forall z_1, z_2 \in \mathbb{C}$,

$$z_1 \bar{\cdot} z_2 = \bar{z_1} \cdot \bar{z_2}.$$

Solution:

Let $z_1 = a + ib$ and $z_2 = c + id$.

L.H.S.:

$$z_1 z_2 = (a + ib)(c + id) = (ac - bd) + i(ad + bc) z_1 \bar{z}_2 = (ac - bd) - i(ad + bc)(\text{i})$$

R.H.S.:

$$\bar{z_1} = a - ib, \bar{z_2} = c - id \bar{z_1} \cdot \bar{z_2} = (a - ib)(c - id) = (ac - bd) - i(ad + bc)(\text{ii})$$

From (i) and (ii):

$$z_1 \bar{z}_2 = \bar{z_1} \cdot \bar{z_2}$$

Hence proved.

Q.7

Find the square root of the following complex numbers:

(i) $-7 - 24i$

Solution:

Let $z = x + iy = -7 - 24i$.
So $x = -7,\ y = -24$.

$$| z | = \sqrt{(-7)^2 + (-24)^2} = \sqrt{49 + 576} = \sqrt{625} = 25$$

Using the square root formula:

$$\sqrt{z} = \pm \left(\sqrt{\frac{| z | + x}{2}} + i \frac{y}{| y |} \sqrt{\frac{| z | - x}{2}} \right) = \pm \left(\sqrt{\frac{25 - 7}{2}} - i \sqrt{\frac{25 + 7}{2}} \right) = \pm \left(\sqrt{9} - i \sqrt{16} \right) = \pm(3 - 4i)$$

Answer: $\sqrt{-7 - 24i} = \pm(3 - 4i)$

(ii) $8 - 6i$

Solution:

Let $z = x + iy = 8 - 6i$.
So $x = 8,\ y = -6$.

$$|\, z \,| = \sqrt{8^2 + (-6)^2} = \sqrt{64 + 36} = \sqrt{100} = 10\sqrt{z} = \pm\left(\sqrt{\frac{10+8}{2}} - i\sqrt{\frac{10-8}{2}}\right) = \pm\left(\sqrt{9} - i\sqrt{1}\right) = \pm(3 - i)$$

Answer: $\sqrt{8 - 6i} = \pm(3 - i)$

(iii) $\mathbf{-15 - 36i}$

Solution:

Let $z = x + iy = -15 - 36i$.
So $x = -15$, $y = -36$.

$$|\, z \,| = \sqrt{(-15)^2 + (-36)^2} = \sqrt{225 + 1296} = \sqrt{1521} = 39\sqrt{z} = \pm\left(\sqrt{\frac{39-15}{2}} - i\sqrt{\frac{39+15}{2}}\right) = \pm\left(\sqrt{12} - i\sqrt{27}\right) = \pm\left(2\sqrt{3} - i\right.$$

Answer: $\sqrt{-15 - 36i} = \pm\sqrt{3}(2 - 3i)$

(iv) $\mathbf{119 + 120i}$

Solution:

Let $z = x + iy = 119 + 120i$.
So $x = 119$, $y = 120$.

$$|\, z \,| = \sqrt{119^2 + 120^2} = \sqrt{14161 + 14400} = \sqrt{28561} = 169\sqrt{z} = \pm\left(\sqrt{\frac{169+119}{2}} + i\sqrt{\frac{169-119}{2}}\right) = \pm\left(\sqrt{144} + i\sqrt{25}\right) = \pm(12$$

Answer: $\sqrt{119 + 120i} = \pm(12 + 5i)$

Q.8

Find the square root of

$$13 - 20\sqrt{3}i$$

and represent it on an **Argand diagram**.

Solution:

Let $z = x + iy = 13 - 20\sqrt{3}i$.
So $x = 13$, $y = -20\sqrt{3}$.

$$|\, z \,| = \sqrt{13^2 + (-20\sqrt{3})^2} = \sqrt{169 + 1200} = \sqrt{1369} = 37\sqrt{z} = \pm\left(\sqrt{\frac{37+13}{2}} - i\sqrt{\frac{37-13}{2}}\right) = \pm\left(\sqrt{25} - i\sqrt{12}\right) = \pm(5 - i2\sqrt{3}$$

Answer: $\sqrt{13 - 20\sqrt{3}i} = \pm(5 - 2\sqrt{3}i)$

Argand Diagram:
The two roots are $5 - 2\sqrt{3}i$ and $-5 + 2\sqrt{3}i$.
Plot them at coordinates $(5, -3.464)$ and $(-5, 3.464)$.

Q.9

Find the real values of x and y if:

$$(-7+i)(x+iy)+(-1-5i)=i(11-i).$$

Solution:

$$(-7+i)(x+iy)+(-1-5i)=i(11-i)-7x-7iy+ix+i^2y-1-5i=11i-i^2-7x-7iy+ix-y-1-5i=11i+1(-7x$$

Comparing:

$$-7x-y-1=1 \Rightarrow -7x-y=2(\text{i})x-7y-5=11 \Rightarrow x-7y=16(\text{ii})$$

From (ii):

$$x=16+7y$$

Substitute in (i):

$$-7(16+7y)-y=2-112-49y-y=2-50y=114y=-\frac{57}{25}$$

Then:

$$x=16+7\left(-\frac{57}{25}\right)=16-\frac{399}{25}=\frac{400-399}{25}=\frac{1}{25}$$

Answer: $x=\frac{1}{25}$, $y=-\frac{57}{25}$

Q.10

Find the real values of x and y:

$$(5-2i)(x+iy)+3=i(11-i)-4i.$$

Solution:

$$(5-2i)(x+iy)+3=i(11-i)-4i5x+5iy-2ix-2i^2y+3=11i-i^2-4i5x+5iy-2ix+2y+3=11i+1-4i(5x+2y$$

Comparing:

$$5x+2y+3=1 \Rightarrow 5x+2y=-2(\text{i})5y-2x=7(\text{ii})$$

Solving (i) and (ii):

Multiply (i) by 5 and (ii) by 2:

$$25x+10y=-10-4x+10y=14$$

Subtract:

$$29x = -24 \Rightarrow x = -\frac{24}{29}$$

Substitute in (i):

$$5\left(-\frac{24}{29}\right) + 2y = -2 - \frac{120}{29} + 2y = -22y = \frac{62}{29}y = \frac{31}{29}$$

Answer: $x = -\frac{24}{29}$, $y = \frac{31}{29}$

Q.11

Find the real values of u and v if:

$$\frac{u-2}{2+i} + \frac{v-3}{2-i} = 4i.$$

Solution:

$$\frac{u-2}{2+i} \cdot \frac{2-i}{2-i} + \frac{v-3}{2-i} \cdot \frac{2+i}{2+i} = 4i\frac{(u-2)(2-i)+(v-3)(2+i)}{5} = 4i(u-2)(2-i)+(v-3)(2+i) = 20i(2u+2v-10)+i(v$$

Comparing:

$$2u + 2v - 10 = 0 \Rightarrow u + v = 5(\text{i})v - u - 1 = 20 \Rightarrow v - u = 21(\text{ii})$$

From (i) and (ii):

$$u = -8, v = 13$$

Answer: $u = -8$, $v = 13$

Q.12

If $z_1 = 4 + 5i$ and $z_2 = \alpha - 2i$, find the real value of α such that

$$Re(z_1 z_2) = 20.$$

Solution:

$$z_1 z_2 = (4+5i)(\alpha - 2i) = 4\alpha - 8i + 5i\alpha - 10i^2 = 4\alpha + 10 + i(5\alpha - 8)Re(z_1 z_2) = 4\alpha + 10 = 204\alpha = 10\alpha = \frac{5}{2}$$

Answer: $\alpha = \frac{5}{2}$