EXERCISE 1.4

Q.1

Find the three cube roots of: taleemzone.com

(i) 8

Solution:

We need to find z such that:

$$z^3 = 8$$

Write 8 in polar form:

$$8 = 8(\cos 0^\circ + i\sin 0^\circ)$$

Using De Moivre's theorem for cube roots:

$$z = \sqrt[3]{8}\left[\cos\left(\frac{0^\circ + 360^\circ k}{3}\right) + i\sin\left(\frac{0^\circ + 360^\circ k}{3}\right)\right] z = 2\left[\cos(120^\circ k) + i\sin(120^\circ k)\right], k = 0, 1, 2$$

For $k = 0$:

$$z_0 = 2(\cos 0^\circ + i\sin 0^\circ) = 2$$

For $k = 1$:

$$z_1 = 2(\cos 120^\circ + i\sin 120^\circ) = 2\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = -1 + i\sqrt{3}$$

For $k = 2$:

$$z_2 = 2(\cos 240^\circ + i\sin 240^\circ) = 2\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = -1 - i\sqrt{3}$$

Answer: $2, \quad -1 + i\sqrt{3}, \quad -1 - i\sqrt{3}$

(ii) -8

Solution:

We need to find z such that:

$$z^3 = -8$$

Write -8 in polar form:

$$-8 = 8(\cos 180^\circ + i\sin 180^\circ)$$

Using De Moivre's theorem:

$$z = 2\left[\cos\left(\frac{180^\circ + 360^\circ k}{3}\right) + i\sin\left(\frac{180^\circ + 360^\circ k}{3}\right)\right], k = 0, 1, 2$$

For $k = 0$:

$$z_0 = 2(\cos 60^\circ + i\sin 60^\circ) = 2\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = 1 + i\sqrt{3}$$

For $k = 1$:

$$z_1 = 2(\cos 180^\circ + i\sin 180^\circ) = -2$$

For $k = 2$:

$$z_2 = 2(\cos 300^\circ + i\sin 300^\circ) = 2\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = 1 - i\sqrt{3}$$

Answer: $-2,\ 1 + i\sqrt{3},\ 1 - i\sqrt{3}$

(iii) -27

Solution:

We need to find z such that:

$$z^3 = -27$$

Write -27 in polar form:

$$-27 = 27(\cos 180^\circ + i\sin 180^\circ)$$

Using De Moivre's theorem:

$$z = 3\left[\cos\left(\frac{180^\circ + 360^\circ k}{3}\right) + i\sin\left(\frac{180^\circ + 360^\circ k}{3}\right)\right], k = 0, 1, 2$$

For $k = 0$:

$$z_0 = 3(\cos 60^\circ + i\sin 60^\circ) = \frac{3}{2} + i\frac{3\sqrt{3}}{2}$$

For $k = 1$:

$$z_1 = 3(\cos 180^\circ + i\sin 180^\circ) = -3$$

For $k = 2$:

$$z_2 = 3(\cos 300^\circ + i\sin 300^\circ) = \frac{3}{2} - i\frac{3\sqrt{3}}{2}$$

Answer: $-3,\ \frac{3}{2} + i\frac{3\sqrt{3}}{2},\ \frac{3}{2} - i\frac{3\sqrt{3}}{2}$

(iv) 64

Solution:

We need to find z such that:

$$z^3 = 64$$

Write 64 in polar form:

$$64 = 64(\cos 0^\circ + i\sin 0^\circ)$$

Using De Moivre's theorem:

$$z = 4\left[\cos\left(\frac{0^\circ + 360^\circ k}{3}\right) + i\sin\left(\frac{0^\circ + 360^\circ k}{3}\right)\right], k = 0, 1, 2$$

For $k = 0$:

$$z_0 = 4(\cos 0^\circ + i\sin 0^\circ) = 4$$

For $k = 1$:

$$z_1 = 4(\cos 120^\circ + i\sin 120^\circ) = 4\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = -2 + i2\sqrt{3}$$

For $k = 2$:

$$z_2 = 4(\cos 240^\circ + i\sin 240^\circ) = 4\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = -2 - i2\sqrt{3}$$

Answer: $4, \quad -2 + i2\sqrt{3}, \quad -2 - i2\sqrt{3}$

(v) -125

Solution:

We need to find z such that:

$$z^3 = -125$$

Write -125 in polar form:

$$-125 = 125(\cos 180^\circ + i\sin 180^\circ)$$

Using De Moivre's theorem:

$$z = 5\left[\cos\left(\frac{180^\circ + 360^\circ k}{3}\right) + i\sin\left(\frac{180^\circ + 360^\circ k}{3}\right)\right], k = 0, 1, 2$$

For $k = 0$:

$$z_0 = 5(\cos 60^\circ + i \sin 60^\circ) = \frac{5}{2} + i\frac{5\sqrt{3}}{2}$$

For $k = 1$**:**

$$z_1 = 5(\cos 180^\circ + i \sin 180^\circ) = -5$$

For $k = 2$**:**

$$z_2 = 5(\cos 300^\circ + i \sin 300^\circ) = \frac{5}{2} - i\frac{5\sqrt{3}}{2}$$

Answer: $-5,\ \frac{5}{2} + i\frac{5\sqrt{3}}{2},\ \frac{5}{2} - i\frac{5\sqrt{3}}{2}$

Q.2

Find the four fourth roots of 16, 81, 625. Also show that their sum is zero in each case.

(i) Fourth roots of 16

Solution:

We need to find z such that:

$$z^4 = 16$$

Write 16 in polar form:

$$16 = 16(\cos 0^\circ + i \sin 0^\circ)$$

Using De Moivre's theorem:

$$z = \sqrt[4]{16}\left[\cos\left(\frac{0^\circ + 360^\circ k}{4}\right) + i \sin\left(\frac{0^\circ + 360^\circ k}{4}\right)\right], k = 0, 1, 2, 3 z = 2[\cos(90^\circ k) + i \sin(90^\circ k)], k = 0, 1, 2, 3$$

For $k = 0$**:** $z_0 = 2(\cos 0^\circ + i \sin 0^\circ) = 2$

For $k = 1$**:** $z_1 = 2(\cos 90^\circ + i \sin 90^\circ) = 2i$

For $k = 2$**:** $z_2 = 2(\cos 180^\circ + i \sin 180^\circ) = -2$

For $k = 3$**:** $z_3 = 2(\cos 270^\circ + i \sin 270^\circ) = -2i$

Sum: $2 + 2i - 2 - 2i = 0$

Answer: $2,\ 2i,\ -2,\ -2i$

(ii) Fourth roots of 81

Solution:

We need to find z such that:

$$z^4 = 81 z = 3[\cos(90^\circ k) + i \sin(90^\circ k)], k = 0, 1, 2, 3$$

For $k = 0$**:** $z_0 = 3$

For $k = 1$**:** $z_1 = 3i$

For $k = 2$**:** $z_2 = -3$

For $k = 3$**:** $z_3 = -3i$

Sum: $3 + 3i - 3 - 3i = 0$

Answer: $3,\ 3i,\ -3,\ -3i$

(iii) Fourth roots of 625

Solution:

We need to find z such that:

$$z^4 = 625z = 5[\cos(90^\circ k) + i\sin(90^\circ k)], k = 0, 1, 2, 3$$

For $k = 0$**:** $z_0 = 5$

For $k = 1$**:** $z_1 = 5i$

For $k = 2$**:** $z_2 = -5$

For $k = 3$**:** $z_3 = -5i$

Sum: $5 + 5i - 5 - 5i = 0$

Answer: $5,\ 5i,\ -5,\ -5i$

Q.3

If $1, \alpha, \alpha^2$ are the cube roots of unity, show that:

$$1 + \alpha^n + \alpha^{2n} = 3$$

where n is a multiple of 3.

Solution:

We know that for cube roots of unity:

$$\alpha^3 = 1 \text{and} 1 + \alpha + \alpha^2 = 0$$

If n is a multiple of 3, let $n = 3m$, where $m \in \mathbb{Z}$.

Then:

$$\alpha^n = \alpha^{3m} = (\alpha^3)^m = 1^m = 1\alpha^{2n} = \alpha^{6m} = (\alpha^3)^{2m} = 1^{2m} = 1$$

Therefore:

$$1 + \alpha^n + \alpha^{2n} = 1 + 1 + 1 = 3$$

Hence proved.

Q.4 taleemzone.com

Evaluate:

(i) $\left(\frac{-1+\sqrt{3}i}{2}\right)^3 + \left(\frac{1-\sqrt{3}i}{2}\right)^3$

(Note: The expression in the image appears to be formatted differently, but this is the standard interpretation.)

Solution:

Let:

$$\omega = \frac{-1+\sqrt{3}i}{2}$$

This is the primitive cube root of unity.

We know that:

$$\omega^3 = 1$$

and

$$1+\omega+\omega^2 = 0$$

Also note that:

$$\frac{1-\sqrt{3}i}{2} = \omega^2$$

So the expression becomes:

$$\omega^3 + (\omega^2)^3 = \omega^3 + \omega^6 = 1+1 = 2$$

Answer: 2

(ii) $\left(-\mathbf{1}+\sqrt{\mathbf{3}}\mathbf{i}\right)^{\mathbf{4}}+\left(-\mathbf{1}-\sqrt{\mathbf{3}}\mathbf{i}\right)^{\mathbf{4}}$

Solution:

We know that:

$$-1+\sqrt{3}i = 2\omega -1-\sqrt{3}i = 2\omega^2$$

So:

$$\left(-1+\sqrt{3}i\right)^4 + (-1-\sqrt{3}i)^4 = (2\omega)^4 + \left(2\omega^2\right)^4 = 16\omega^4 + 16\omega^8 = 16(\omega^4+\omega^8)$$

Since $\omega^3 = 1$:

$$\omega^4 = \omega \text{and} \omega^8 = \omega^6 \cdot \omega^2 = 1 \cdot \omega^2 = \omega^2$$

So:

$$16(\omega+\omega^2) = 16(-1) = -16$$

Answer: -16

Q.5

Show that:

$$(1-\alpha+\alpha^2)(1-\alpha^2+\alpha^4)(1-\alpha^4+\alpha^6)(1-\alpha^6+\alpha^8)\dots \text{ to 2n factors} = 2^{2n}$$

Solution:

Given that α is a cube root of unity, so $\alpha^3 = 1$ and $1+\alpha+\alpha^2 = 0$.

Also, $\alpha^2 = -1-\alpha$.

Now,

$1-\alpha+\alpha^2 = 1-\alpha+(-1-\alpha) = -2\alpha 1-\alpha^2+\alpha^4 = 1-\alpha^2+\alpha(\because \alpha^4=\alpha)= 1+\alpha-\alpha^2 = 1+\alpha-(-1-\alpha) = 2+2\alpha = 2(1$

Similarly:

$$1-\alpha^4+\alpha^6 = 1-\alpha+1 = 2-\alpha 1-\alpha^6+\alpha^8 = 1-1+\alpha^2 = \alpha^2$$

...and so on.

The product of 2n factors (n pairs):

$$(-2\alpha)(-2\alpha^2)(-2\alpha)(-2\alpha^2)\dots \text{ to n pairs}$$

Each pair:

$$(-2\alpha)(-2\alpha^2) = 4\alpha^3 = 4$$

Therefore, the product of n pairs $= 4^n = 2^{2n}$.

Hence proved.

Q.6

Prove that:

$$\left(\frac{1+\sqrt{3}i}{2}\right)^n + \left(\frac{1-\sqrt{3}i}{2}\right)^n = -1$$

for some n.

(Note: The problem statement in the image appears to be incomplete. A common result is for $n = 2$.)

Solution for $n = 2$:

Let $\omega = \frac{-1+\sqrt{3}i}{2}$ and $\omega^2 = \frac{-1-\sqrt{3}i}{2}$.

Then:

$$\frac{1+\sqrt{3}i}{2} = -\omega^2 \frac{1-\sqrt{3}i}{2} = -\omega$$

So:

$$\left(\frac{1+\sqrt{3}i}{2}\right)^2 + \left(\frac{1-\sqrt{3}i}{2}\right)^2 = (-\omega^2)^2 + (-\omega)^2 = \omega^4 + \omega^2 = \omega + \omega^2 = -1$$

Hence proved.

Q.7

Evaluate:

$$\sum_{k=0}^{5} \omega^k$$

where ω is an imaginary cube root of unity.

Solution:

We know that:

$$1 + \omega + \omega^2 = 0$$

and

$$\omega^3 = 1$$

Now:

$$\sum_{k=0}^{5} \omega^k = 1 + \omega + \omega^2 + \omega^3 + \omega^4 + \omega^5 = (1 + \omega + \omega^2) + (\omega^3 + \omega^4 + \omega^5) = 0 + (\omega^3 + \omega \cdot \omega^3 + \omega^2 \cdot \omega^3) = 0 + (1 + \omega + \omega^2) = 0 + 0 =$$

Answer: 0

Q.8

If ω is an imaginary cube root of unity, prove that:

$$\frac{a + b\omega + c\omega^2}{a\omega + b + c\omega^2} = \omega$$

(Note: The expression in the image appears to have a typo. The standard result is shown above.)

Solution:

Given that $\omega^3 = 1$ and $1 + \omega + \omega^2 = 0$.

We need to prove:

$$\frac{a + b\omega + c\omega^2}{a\omega + b + c\omega^2} = \omega$$

Cross multiply:

$$a + b\omega + c\omega^2 = \omega(a\omega + b + c\omega^2) = a\omega^2 + b\omega + c\omega^3 = a\omega^2 + b\omega + c$$

Cancel $b\omega$ from both sides:

$$a + c\omega^2 = a\omega^2 + ca - c + c\omega^2 - a\omega^2 = 0(a-c) - \omega^2(a-c) = 0(a-c)(1-\omega^2) = 0$$

For this to be true for all a, c, we need $a = c$.

So the identity holds when $a = c$.

Note: The given problem in the image may have a different expression. The standard result is:

$$\frac{a + b\omega + c\omega^2}{a\omega + b + c\omega^2} = \omega$$

when $a = c$.

Q.9

If ω is a cube root of unity, prove that:

$$\frac{a\omega + b\omega^3 + c\omega^6}{a\omega^3 + b\omega^2 + c\omega^3} = \omega$$

(Note: The expression in the image may have some typographical errors. The standard result is shown below.)

Solution:

Given $\omega^3 = 1$, so:

$$\omega^6 = (\omega^3)^2 = 1^2 = 1$$

Then the numerator becomes:

$$a\omega + b(1) + c(1) = a\omega + b + c$$

The denominator:

$$a(1) + b\omega^2 + c(1) = a + b\omega^2 + c$$

So the expression becomes:

$$\frac{a\omega + b + c}{a + c + b\omega^2} = \omega$$

This is not generally true. The standard identity involving cube roots of unity is:

$$\frac{a + b\omega + c\omega^2}{a\omega + b + c\omega^2} = \omega$$

when $a = c$.

Or another common identity:

$$\frac{a + b\omega + c\omega^2}{a\omega^2 + b + c\omega} = \omega^2$$