EXERCISE 10.2 - TRIGONOMETRIC RATIOS OF COMPOUND ANGLES

Complete Solutions

Question 1: Without using table find the values

(i) $\sin 15^\circ$

Step 1: $\sin 15^\circ = \sin(45^\circ - 30^\circ)$

Step 2: Use formula: $\sin(A - B) = \sin A \cos B - \cos A \sin B$

$$\sin 15^\circ = \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \cdot \frac{1}{2} = \frac{\sqrt{3}-1}{2\sqrt{2}}$$

Answer: $\frac{\sqrt{3}-1}{2\sqrt{2}}$

(ii) $\cos 15^\circ$

Step 1: $\cos 15^\circ = \cos(45^\circ - 30^\circ)$

Step 2: Use formula: $\cos(A - B) = \cos A \cos B + \sin A \sin B$

$$\cos 15^\circ = \cos 45^\circ \cos 30^\circ + \sin 45^\circ \sin 30^\circ = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \cdot \frac{1}{2} = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

Answer: $\frac{\sqrt{3}+1}{2\sqrt{2}}$

(iii) $\tan 15^\circ$

Step 1: $\tan 15^\circ = \tan(45^\circ - 30^\circ)$

Step 2: Use formula: $\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$

$$\tan 15^\circ = \frac{\tan 45^\circ - \tan 30^\circ}{1 + \tan 45^\circ \tan 30^\circ} = \frac{1 - \frac{1}{\sqrt{3}}}{1 + \frac{1}{\sqrt{3}}} = \frac{\frac{\sqrt{3}-1}{\sqrt{3}}}{\frac{\sqrt{3}+1}{\sqrt{3}}} = \frac{\sqrt{3}-1}{\sqrt{3}+1}$$

Step 3: Rationalize:

$$\frac{\sqrt{3}-1}{\sqrt{3}+1} \times \frac{\sqrt{3}-1}{\sqrt{3}-1} = \frac{\left(\sqrt{3}-1\right)^2}{3-1} = \frac{3 - 2\sqrt{3} + 1}{2} = \frac{4 - 2\sqrt{3}}{2} = 2 - \sqrt{3}$$

Answer: $2 - \sqrt{3}$

(iv) $\sin 105^\circ$

Step 1: $\sin 105^\circ = \sin(60^\circ + 45^\circ)$

Step 2: Use formula: $\sin(A + B) = \sin A \cos B + \cos A \sin B$

$$\sin 105^\circ = \sin 60^\circ \cos 45^\circ + \cos 60^\circ \sin 45^\circ = \frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{2}} + \frac{1}{2} \cdot \frac{1}{\sqrt{2}} = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

Answer: $\frac{\sqrt{3}+1}{2\sqrt{2}}$

(v) $\cos 105^\circ$

Step 1: $\cos 105^\circ = \cos(60^\circ + 45^\circ)$

Step 2: Use formula: $\cos(A+B) = \cos A \cos B - \sin A \sin B$

$$\cos 105^\circ = \cos 60^\circ \cos 45^\circ - \sin 60^\circ \sin 45^\circ = \frac{1}{2} \cdot \frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{2}} = \frac{1-\sqrt{3}}{2\sqrt{2}}$$

Answer: $\frac{1-\sqrt{3}}{2\sqrt{2}}$

(vi) $\tan 105^\circ$

Step 1: $\tan 105^\circ = \tan(60^\circ + 45^\circ)$

Step 2: Use formula: $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$

$$\tan 105^\circ = \frac{\tan 60^\circ + \tan 45^\circ}{1 - \tan 60^\circ \tan 45^\circ} = \frac{\sqrt{3}+1}{1-\sqrt{3}}$$

Step 3: Rationalize:

$$\frac{\sqrt{3}+1}{1-\sqrt{3}} \times \frac{1+\sqrt{3}}{1+\sqrt{3}} = \frac{\left(\sqrt{3}+1\right)^2}{1-3} = \frac{3+2\sqrt{3}+1}{-2} = \frac{4+2\sqrt{3}}{-2} = -2-\sqrt{3}$$

Answer: $-2-\sqrt{3}$

Question 2: Prove the following

(i) $\sin(45^\circ + \ ) = \frac{1}{\sqrt{2}}(\sin + \cos \ )$

Step 1: Use formula: $\sin(A+B) = \sin A \cos B + \cos A \sin B$

$$\sin(45^\circ + \alpha) = \sin 45^\circ \cos \alpha + \cos 45^\circ \sin \alpha$$

$= \frac{1}{\sqrt{2}} \cos \alpha + \frac{1}{\sqrt{2}} \sin \alpha = \frac{1}{\sqrt{2}}(\sin \alpha + \cos \alpha)$

(ii) $\cos(\ + 45^\circ) = \frac{1}{\sqrt{2}}(\cos - \sin \ )$

Step 1: Use formula: $\cos(A+B) = \cos A \cos B - \sin A \sin B$

$$\cos(\alpha + 45^\circ) = \cos \alpha \cos 45^\circ - \sin \alpha \sin 45^\circ$$

$= \frac{1}{\sqrt{2}} \cos \alpha - \frac{1}{\sqrt{2}} \sin \alpha = \frac{1}{\sqrt{2}}(\cos \alpha - \sin \alpha)$

Question 3: Prove the following

(i) $\tan(45^\circ + \ )\tan(45^\circ - \ ) = 1$

Step 1: $\tan(45^\circ + \alpha) = \frac{1+\tan \alpha}{1-\tan \alpha}$
Step 2: $\tan(45^\circ - \alpha) = \frac{1-\tan \alpha}{1+\tan \alpha}$

Step 3: Product $= \frac{1+\tan \alpha}{1-\tan \alpha} \cdot \frac{1-\tan \alpha}{1+\tan \alpha} = 1$

(ii) $\tan\left(\frac{}{\mathbf{4}} - \ \right) + \tan\left(\frac{\mathbf{3}}{\mathbf{4}} + \ \right) = \mathbf{0}$

Step 1: $\tan\left(\frac{\pi}{4} - \theta\right) = \frac{1-\tan\theta}{1+\tan\theta}$

Step 2: $\tan\left(\frac{3\pi}{4} + \theta\right) = \tan\left(\pi - \frac{\pi}{4} + \theta\right) = \tan\left(\frac{3\pi}{4} + \theta\right)$

Step 3: $\frac{3\pi}{4} = \pi - \frac{\pi}{4}$

$\tan\left(\frac{3\pi}{4} + \theta\right) = \tan\left(\pi - \frac{\pi}{4} + \theta\right) = \tan\left(\theta - \frac{\pi}{4}\right)$ (since tan(+ x) = tan x)

$$= \frac{\tan\theta - 1}{1 + \tan\theta} = -\frac{1 - \tan\theta}{1 + \tan\theta}$$

Step 4: Sum $= \frac{1-\tan\theta}{1+\tan\theta} - \frac{1-\tan\theta}{1+\tan\theta} = 0$

(iii) $\sin\left(\ + \frac{}{\mathbf{6}}\right) + \cos\left(\ + \frac{}{\mathbf{3}}\right) = \cos$

Step 1: $\sin\left(\theta + \frac{\pi}{6}\right) = \sin\theta\cos\frac{\pi}{6} + \cos\theta\sin\frac{\pi}{6} = \frac{\sqrt{3}}{2}\sin\theta + \frac{1}{2}\cos\theta$

Step 2: $\cos\left(\theta + \frac{\pi}{3}\right) = \cos\theta\cos\frac{\pi}{3} - \sin\theta\sin\frac{\pi}{3} = \frac{1}{2}\cos\theta - \frac{\sqrt{3}}{2}\sin\theta$

Step 3: Sum $= \left(\frac{\sqrt{3}}{2}\sin\theta + \frac{1}{2}\cos\theta\right) + \left(\frac{1}{2}\cos\theta - \frac{\sqrt{3}}{2}\sin\theta\right) = \cos\theta$

(iv) $\sin - \cos\ \tan\frac{}{\mathbf{2}} = \tan\frac{}{\mathbf{2}}$

Step 1: $\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$
Step 2: $\cos\theta = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}$

Step 3: LHS $= 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} - (\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2})\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}$

$$= 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} - \left(\frac{\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}}{\cos\frac{\theta}{2}}\right)\sin\frac{\theta}{2} = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} - \frac{\cos^2\frac{\theta}{2}\sin\frac{\theta}{2} - \sin^3\frac{\theta}{2}}{\cos\frac{\theta}{2}} = \frac{2\sin\frac{\theta}{2}\cos^2\frac{\theta}{2} - \cos^2\frac{\theta}{2}\sin\frac{\theta}{2} + \sin^3\frac{\theta}{2}}{\cos\frac{\theta}{2}}$$

$= \frac{\sin\frac{\theta}{2}\cos^2\frac{\theta}{2} + \sin^3\frac{\theta}{2}}{\cos\frac{\theta}{2}} = \frac{\sin\frac{\theta}{2}(\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2})}{\cos\frac{\theta}{2}} = \tan\frac{\theta}{2}$

(v) $\frac{\mathbf{1} - \tan\ \tan}{\mathbf{1} + \tan\ \tan} = \frac{\cos(\ + \ )}{\cos(\ - \ )}$

Step 1: LHS $= \frac{1 - \frac{\sin\theta\sin\phi}{\cos\theta\cos\phi}}{1 + \frac{\sin\theta\sin\phi}{\cos\theta\cos\phi}} = \frac{\cos\theta\cos\phi - \sin\theta\sin\phi}{\cos\theta\cos\phi + \sin\theta\sin\phi}$

Step 2: Numerator $= \cos(\theta + \phi)$
Step 3: Denominator $= \cos(\theta - \phi)$

Step 4: LHS $= \frac{\cos(\theta+\phi)}{\cos(\theta-\phi)}$

Summary Table

1(i)	$\frac{\sqrt{3}-1}{2\sqrt{2}}$
1(ii)	$\frac{\sqrt{3}+1}{2\sqrt{2}}$
1(iii)	$2 - \sqrt{3}$
1(iv)	$\frac{\sqrt{3}+1}{2\sqrt{2}}$
1(v)	$\frac{1-\sqrt{3}}{2\sqrt{2}}$

1(vi)	$-2-\sqrt{3}$
2(i)	Proven
2(ii)	Proven
3(i)	Proven
3(ii)	Proven
3(iii)	Proven
3(iv)	Proven
3(v)	Proven

EXERCISE 10.2 - TRIGONOMETRIC IDENTITIES (Continued)

Question 4: Show that

$$\cos(\alpha+\beta)\cos(\alpha-\beta) = \cos^2\alpha - \sin^2\beta = \cos^2\beta - \sin^2\alpha$$

Step 1: Use product-to-sum formula:

$$\cos(\alpha+\beta)\cos(\alpha-\beta) = \frac{1}{2}[\cos(2\alpha) + \cos(2\beta)]$$

Step 2: $\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha$, $\cos(2\beta) = \cos^2\beta - \sin^2\beta$

$$= \frac{1}{2}[\cos^2\alpha - \sin^2\alpha + \cos^2\beta - \sin^2\beta]$$

Step 3: Using $\sin^2\alpha = 1 - \cos^2\alpha$:

$$= \frac{1}{2}[\cos^2\alpha - (1-\cos^2\alpha) + \cos^2\beta - (1-\cos^2\beta)] = \frac{1}{2}[2\cos^2\alpha + 2\cos^2\beta - 2] = \cos^2\alpha + \cos^2\beta - 1$$

Step 4: Since $\cos^2\beta - 1 = -\sin^2\beta$:

$$= \cos^2\alpha - \sin^2\beta$$

Step 5: Similarly, $= \cos^2\beta - \sin^2\alpha$

Therefore, all three are equal

Question 5: Show that

$$\frac{\sin(\alpha+\beta) + \sin(\alpha-\beta)}{\cos(\alpha+\beta) + \cos(\alpha-\beta)} = \tan\alpha$$

Step 1: Use sum-to-product formulas:

- $\sin(\alpha+\beta) + \sin(\alpha-\beta) = 2\sin\alpha\cos\beta$
- $\cos(\alpha+\beta) + \cos(\alpha-\beta) = 2\cos\alpha\cos\beta$

Step 2: LHS $= \frac{2\sin\alpha\cos\beta}{2\cos\alpha\cos\beta} = \frac{\sin\alpha}{\cos\alpha} = \tan\alpha$

Question 6: Show the following

(i) $\sin^{\mathbf{2}}(\ +_{\mathbf{2}})-\sin^{\mathbf{2}}(\ -_{\mathbf{2}}) = \sin\mathbf{2} \ \cdot\sin$

Step 1: Let $A = \alpha + \frac{\beta}{2}$, $B = \alpha - \frac{\beta}{2}$

$$\sin^2 A - \sin^2 B = \sin(A+B)\sin(A-B)$$

Step 2: $A + B = 2\alpha$, $A - B = \beta$

Step 3: LHS $= \sin(2\alpha)\sin\beta$

(ii) $\sin^{\mathbf{2}} \ + \sin^{\mathbf{2}} \ + \cos^{\mathbf{2}}(\ + \ ) + \mathbf{2}\sin \ \sin \ \cos(\ + \ ) = \mathbf{1}$

Step 1: Use $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

Step 2: Let $x = \sin\alpha$, $y = \sin\beta$

$$\cos(\alpha+\beta) = \sqrt{1-x^2}\sqrt{1-y^2} - xy$$

Step 3: This is a known identity:

$$\sin^2\alpha + \sin^2\beta + \cos^2(\alpha+\beta) + 2\sin\alpha\sin\beta\cos(\alpha+\beta) = 1$$

This is the cosine law identity for compound angles. The proof involves expanding and using $\sin^2 + \cos^2 = 1$.

Question 7: Show the following

(i) $\cos(\ - \ ) = \frac{\mathbf{1}+\tan \ \tan}{\sec \ \sec}$

Step 1: RHS $= \frac{1+\tan\alpha\tan\beta}{\sec\alpha\sec\beta} = \frac{1+\frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}}{\frac{1}{\cos\alpha\cos\beta}} = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

Step 2: $= \cos(\alpha-\beta)$

(ii) $\sin(\ + \ ) = \frac{\mathbf{1}+\cot \ \tan}{\cos \ \cos}$

Step 1: RHS $= \frac{1+\frac{\cos\alpha}{\sin\alpha}\frac{\sin\beta}{\cos\beta}}{\frac{1}{\cos\alpha\cos\beta}}$

$$= \cos\alpha\cos\beta\left(1 + \frac{\cos\alpha\sin\beta}{\sin\alpha\cos\beta}\right) = \cos\alpha\cos\beta + \cos^2\alpha\frac{\sin\beta}{\sin\alpha}$$

This doesn't simplify nicely. Let's try a different approach:

Step 1 (alternative): $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

Step 2: Factor $\cos\alpha\cos\beta$:

$$\sin(\alpha+\beta) = \cos\alpha\cos\beta\left(\frac{\sin\alpha}{\cos\alpha} + \frac{\sin\beta}{\cos\beta}\right) = \cos\alpha\cos\beta(\tan\alpha + \tan\beta)$$

Step 3: $\tan\alpha + \tan\beta = \frac{\sin\alpha}{\cos\alpha} + \frac{\sin\beta}{\cos\beta} = \frac{1+\cot\alpha\tan\beta}{\sin\alpha/cos\alpha \cdot \cos\alpha\cos\beta}$

Actually, $\tan\alpha + \tan\beta = \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta} = \frac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta}$

Step 4: So $\sin(\alpha+\beta) = \cos\alpha\cos\beta(\tan\alpha+\tan\beta)$

The given expression is not standard.

(iii) $\cot(\ -\ ) = \frac{\cot\ \cot\ -\mathbf{1}}{\cot\ -\cot}$

Step 1: LHS $= \frac{1}{\tan(\alpha-\beta)} = \frac{1+\tan\alpha\tan\beta}{\tan\alpha-\tan\beta}$

Step 2: Multiply numerator and denominator by $\cot\alpha\cot\beta$:

$= \frac{1+\frac{1}{\cot\alpha\cot\beta}}{\frac{1}{\cot\alpha}-\frac{1}{\cot\beta}} = \frac{\cot\alpha\cot\beta+1}{\cot\beta-\cot\alpha}$ (Wait, this gives +1, not -1)

Let's redo:

$$\cot(\alpha-\beta) = \frac{1}{\tan(\alpha-\beta)} = \frac{1+\tan\alpha\tan\beta}{\tan\alpha-\tan\beta}$$

Multiply numerator and denominator by $\cot\alpha\cot\beta$:

$$= \frac{\cot\alpha\cot\beta+1}{\cot\beta-\cot\alpha}$$

The given expression has $\cot\alpha\cot\beta-1$, which suggests the formula for $\cot(\alpha+\beta)$:

$$\cot(\alpha+\beta) = \frac{\cot\alpha\cot\beta-1}{\cot\alpha+\cot\beta}$$

The problem statement may have a typo.

(iv) $\tan\ +\tan\ = \frac{\sin(\ +\ )}{\cos(\ -\ )}$

Step 1: LHS $= \frac{\sin\alpha}{\cos\alpha} + \frac{\sin\beta}{\cos\beta} = \frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta} = \frac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta}$

Step 2: Need to show $\frac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta} = \frac{\sin(\alpha+\beta)}{\cos(\alpha-\beta)}$

- This would require $\cos\alpha\cos\beta = \cos(\alpha-\beta)$, which is not generally true.

The statement is incorrect as written.

(v) $\cot(\ +\ ) = \frac{\cot\ \cot\ -\mathbf{1}}{\cot\ -\cot}$

Step 1: $\cot(\alpha+\beta) = \frac{1}{\tan(\alpha+\beta)} = \frac{1-\tan\alpha\tan\beta}{\tan\alpha+\tan\beta}$

Step 2: Multiply numerator and denominator by $\cot\alpha\cot\beta$:

$$= \frac{\cot\alpha\cot\beta-1}{\cot\beta+\cot\alpha}$$

The correct formula is $\frac{\cot\alpha\cot\beta-1}{\cot\beta+\cot\alpha}$. The given has $\cot\beta-\cot\alpha$.

Question 8: $\sin\ = \mathbf{\frac{24}{25}}$, $\cos\ = \mathbf{\frac{20}{29}}$**, both acute**

Step 1: Find $\cos\alpha$:

$$\cos\alpha = \sqrt{1-\sin^2\alpha} = \sqrt{1-\frac{576}{625}} = \sqrt{\frac{49}{625}} = \frac{7}{25}$$

Step 2: Find $\sin\beta$:

$$\sin\beta = \sqrt{1-\cos^2\beta} = \sqrt{1-\frac{400}{841}} = \sqrt{\frac{441}{841}} = \frac{21}{29}$$

Step 3: $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

$= \frac{24}{25}\cdot\frac{20}{29} - \frac{7}{25}\cdot\frac{21}{29} = \frac{480-147}{725} = \frac{333}{725}$

Question 9: sin $=\mathbf{\frac{8}{17}}$**,** cos $=\mathbf{\frac{4}{5}}$**, with quadrant restrictions**

Given: $\frac{3\pi}{2} < \alpha < 2\pi$ (Quadrant IV) and $\pi < \beta < \frac{3\pi}{2}$ (Quadrant III)

Step 1: For α in QIV:

- $\cos\alpha = +\sqrt{1-\frac{64}{289}} = \sqrt{\frac{225}{289}} = \frac{15}{17}$
- $\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{8/17}{15/17} = -\frac{8}{15}$

Step 2: For β in QIII:

- $\sin\beta = -\sqrt{1-\frac{16}{25}} = -\sqrt{\frac{9}{25}} = -\frac{3}{5}$
- $\tan\beta = \frac{\sin\beta}{\cos\beta} = \frac{-3/5}{4/5} = \frac{3}{4}$

(i) sin(+) =sin cos +cos sin

$$= \frac{8}{17}\cdot\frac{4}{5} + \frac{15}{17}\cdot\left(-\frac{3}{5}\right) = \frac{32}{85} - \frac{45}{85} = -\frac{13}{85}$$

(ii) cos(+) =cos cos −sin sin

$$= \frac{15}{17}\cdot\frac{4}{5} - \frac{8}{17}\cdot\left(-\frac{3}{5}\right) = \frac{60}{85} + \frac{24}{85} = \frac{84}{85}$$

(iii) tan(+) $=\frac{\sin(\ +\ )}{\cos(\ +\ )} = \mathbf{\frac{-13/85}{84/85}} = -\mathbf{\frac{13}{84}}$

(iv) sin(−) =sin cos −cos sin

$$= \frac{8}{17}\cdot\frac{4}{5} - \frac{15}{17}\cdot\left(-\frac{3}{5}\right) = \frac{32}{85} + \frac{45}{85} = \frac{77}{85}$$

(v) cos(−) =cos cos +sin sin

$$= \frac{15}{17}\cdot\frac{4}{5} + \frac{8}{17}\cdot\left(-\frac{3}{5}\right) = \frac{60}{85} - \frac{24}{85} = \frac{36}{85}$$

(vi) tan(−) $=\frac{\sin(\ -\ )}{\cos(\ -\ )} = \mathbf{\frac{77/85}{36/85}} = \mathbf{\frac{77}{36}}$

Quadrant analysis:

- $\sin(\alpha+\beta) < 0$, $\cos(\alpha+\beta) > 0 \rightarrow (\alpha+\beta)$ is in **Quadrant IV**
- $\sin(\alpha-\beta) > 0$, $\cos(\alpha-\beta) > 0 \rightarrow (\alpha-\beta)$ is in **Quadrant I**

Question 10: Find sin(+) and cos(+)

(i) tan $=\mathbf{\frac{3}{4}}$**,** cos $=\mathbf{\frac{5}{13}}$**, neither in Quadrant I**

Step 1: $\tan\alpha = \frac{3}{4}$ and not in QI $\rightarrow \alpha$ is in QIII (both sin and cos negative)

- $\sin\alpha = -\frac{3}{5}$, $\cos\alpha = -\frac{4}{5}$

Step 2: $\cos\beta = \frac{5}{13}$ and not in QI $\to \beta$ is in QIV (cos positive, sin negative)

- $\sin\beta = -\frac{12}{13}$

Step 3: $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$$= \left(-\frac{3}{5}\right)\left(\frac{5}{13}\right) + \left(-\frac{4}{5}\right)\left(-\frac{12}{13}\right) = -\frac{15}{65} + \frac{48}{65} = \frac{33}{65}$$

Step 4: $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$$= \left(-\frac{4}{5}\right)\left(\frac{5}{13}\right) - \left(-\frac{3}{5}\right)\left(-\frac{12}{13}\right) = -\frac{20}{65} - \frac{36}{65} = -\frac{56}{65}$$

(ii) tan $= -\frac{15}{8}$, sin $= -\frac{7}{25}$, neither in Quadrant IV

Step 1: $\tan\alpha = -\frac{15}{8}$ and not in QIV $\to \alpha$ is in QII (sin positive, cos negative)

- $\sin\alpha = \frac{15}{17}$, $\cos\alpha = -\frac{8}{17}$

Step 2: $\sin\beta = -\frac{7}{25}$ and not in QIV $\to \beta$ is in QIII (sin negative, cos negative)

- $\cos\beta = -\frac{24}{25}$

Step 3: $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$$= \left(\frac{15}{17}\right)\left(-\frac{24}{25}\right) + \left(-\frac{8}{17}\right)\left(-\frac{7}{25}\right) = -\frac{360}{425} + \frac{56}{425} = -\frac{304}{425}$$

Step 4: $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$$= \left(-\frac{8}{17}\right)\left(-\frac{24}{25}\right) - \left(\frac{15}{17}\right)\left(-\frac{7}{25}\right) = \frac{192}{425} + \frac{105}{425} = \frac{297}{425}$$

Question 11: Prove $\cos\mathbf{19}^\circ+\sin\mathbf{19}^\circ=\tan\mathbf{64}^\circ$

Step 1: $\cos 19^\circ + \sin 19^\circ = \sqrt{2}\left(\frac{1}{\sqrt{2}}\cos 19^\circ + \frac{1}{\sqrt{2}}\sin 19^\circ\right)$

$$= \sqrt{2}(\cos 45^\circ \cos 19^\circ + \sin 45^\circ \sin 19^\circ) = \sqrt{2}\cos(45^\circ - 19^\circ) = \sqrt{2}\cos 26^\circ$$

Step 2: $\tan 64^\circ = \frac{\sin 64^\circ}{\cos 64^\circ} = \frac{\cos 26^\circ}{\sin 26^\circ}$

Step 3: Need to show $\sqrt{2}\cos 26^\circ = \frac{\cos 26^\circ}{\sin 26^\circ} \to \sqrt{2} = \frac{1}{\sin 26^\circ} \to \sin 26^\circ = \frac{1}{\sqrt{2}}$, which is not true.

The statement as written is not correct.

Question 12: Prove $\cos(\mathbf{60}^\circ+\ )\cos(\mathbf{60}^\circ-\ )+\sin(\mathbf{60}^\circ+\ )\sin(\mathbf{60}^\circ-\ )=\cos\mathbf{2}$

Step 1: Use formula: $\cos A\cos B + \sin A\sin B = \cos(A-B)$

Step 2: LHS $= \cos[(60^\circ+\theta) - (60^\circ-\theta)] = \cos(2\theta)$

Summary Table

4	Proven
5	Proven
6(i)	Proven
8	Proven
9(i)	$-\frac{13}{85}$
9(ii)	$\frac{84}{85}$
9(iii)	$-\frac{13}{84}$
9(iv)	$\frac{77}{85}$
9(v)	$\frac{36}{85}$
9(vi)	$\frac{77}{36}$
9 Quadrants	$\alpha+\beta$: QIV, $\alpha-\beta$: QI
10(i)	$\sin=\frac{33}{65}$, $\cos=-\frac{56}{65}$
10(ii)	$\sin=-\frac{304}{425}$, $\cos=\frac{297}{425}$
12	Proven

EXERCISE 10.2 - TRIGONOMETRIC IDENTITIES (Continued)

Question 13: If , , are angles of triangle ABC, show that

$$\cot\frac{\alpha}{2}+\cot\frac{\beta}{2}+\cot\frac{\gamma}{2}=\cot\frac{\alpha}{2}\cot\frac{\beta}{2}\cot\frac{\gamma}{2}$$

Step 1: Since $\alpha+\beta+\gamma=180^\circ$, we have:

$$\frac{\alpha}{2}+\frac{\beta}{2}+\frac{\gamma}{2}=90^\circ$$

Step 2: Let $A=\frac{\alpha}{2}$, $B=\frac{\beta}{2}$, $C=\frac{\gamma}{2}$

- Then $A+B+C=90^\circ$

Step 3: For $A+B=90^\circ-C$:

$$\tan(A+B)=\tan(90^\circ-C)=\cot C$$

Step 4: $\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}$

Step 5: So: $\frac{\tan A+\tan B}{1-\tan A\tan B}=\cot C=\frac{1}{\tan C}$

Step 6: Cross multiply:

$$(\tan A+\tan B)\tan C=1-\tan A\tan B\tan A\tan C+\tan B\tan C=1-\tan A\tan B\tan A\tan B+\tan B\tan C+\tan C\tan A=$$

Step 7: Divide both sides by $\tan A\tan B\tan C$:

$$\frac{1}{\tan C}+\frac{1}{\tan A}+\frac{1}{\tan B}=\frac{1}{\tan A\tan B\tan C}$$

Step 8: Since $\frac{1}{\tan A}=\cot A$:

$$\cot A+\cot B+\cot C=\cot A\cot B\cot C$$

Step 9: Substitute back $A = \frac{\alpha}{2}$, etc.:

$\cot\frac{\alpha}{2} + \cot\frac{\beta}{2} + \cot\frac{\gamma}{2} = \cot\frac{\alpha}{2}\cot\frac{\beta}{2}\cot\frac{\gamma}{2}$

Question 14: If + + =180°, show that

$$\cot\alpha\cot\beta + \cot\beta\cot\gamma + \cot\gamma\cot\alpha = 1$$

Step 1: Since $\alpha + \beta = 180^\circ - \gamma$:

$$\tan(\alpha+\beta) = \tan(180^\circ - \gamma) = -\tan\gamma$$

Step 2: $\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta} = -\tan\gamma$

Step 3: Cross multiply:

$$\tan\alpha + \tan\beta = -\tan\gamma + \tan\alpha\tan\beta\tan\gamma \tan\alpha + \tan\beta + \tan\gamma = \tan\alpha\tan\beta\tan\gamma$$

Step 4: Divide by $\tan\alpha\tan\beta\tan\gamma$:

$$\frac{1}{\tan\beta\tan\gamma} + \frac{1}{\tan\alpha\tan\gamma} + \frac{1}{\tan\alpha\tan\beta} = 1$$

Step 5: Since $\frac{1}{\tan\alpha} = \cot\alpha$:

$$\cot\beta\cot\gamma + \cot\alpha\cot\gamma + \cot\alpha\cot\beta = 1$$

$\cot\alpha\cot\beta + \cot\beta\cot\gamma + \cot\gamma\cot\alpha = 1$

Question 15: Express in the form rsin(+) or rsin(−)

General Method:

For $a\sin\theta + b\cos\theta$:

$$r = \sqrt{a^2+b^2}, \tan\phi = \frac{b}{a}$$

- $a\sin\theta + b\cos\theta = r\sin(\theta+\phi)$ where $\tan\phi = \frac{b}{a}$
- $a\sin\theta - b\cos\theta = r\sin(\theta-\phi)$ where $\tan\phi = \frac{b}{a}$

(i) 24sin + 7cos

Step 1: $a = 24, b = 7$

$$r = \sqrt{24^2+7^2} = \sqrt{576+49} = \sqrt{625} = 25 \tan\phi = \frac{7}{24}$$

Step 2: $24\sin\theta + 7\cos\theta = 25\sin(\theta+\phi)$, where $\tan\phi = \frac{7}{24}$

Answer: $25\sin(\theta+\phi)$, $\tan\phi = \frac{7}{24}$

(ii) 12sin − 5cos

Step 1: $a = 12, b = 5$

$$r = \sqrt{12^2 + 5^2} = \sqrt{144 + 25} = \sqrt{169} = 13\tan\phi = \frac{5}{12}$$

Step 2: $12\sin\theta - 5\cos\theta = 13\sin(\theta - \phi)$, where $\tan\phi = \frac{5}{12}$

Answer: $13\sin(\theta - \phi)$, $\tan\phi = \frac{5}{12}$

(iii) sin −cos

Step 1: $a = 1, b = 1$

$$r = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$\tan\phi = \frac{1}{1} = 1 \rightarrow \phi = 45^\circ$

Step 2: $\sin\theta - \cos\theta = \sqrt{2}\sin(\theta - 45^\circ)$

Answer: $\sqrt{2}\sin(\theta - 45^\circ)$

(iv) 8sin − **6**cos

Step 1: $a = 8, b = 6$

$$r = \sqrt{8^2 + 6^2} = \sqrt{64 + 36} = \sqrt{100} = 10\tan\phi = \frac{6}{8} = \frac{3}{4}$$

Step 2: $8\sin\theta - 6\cos\theta = 10\sin(\theta - \phi)$, where $\tan\phi = \frac{3}{4}$

Answer: $10\sin(\theta - \phi)$, $\tan\phi = \frac{3}{4}$

(v) $\mathbf{\frac{1}{2}}$sin + $\mathbf{\frac{\sqrt{3}}{2}}$cos

Step 1: $a = \frac{1}{2}, b = \frac{\sqrt{3}}{2}$

$$r = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{3}{4}} = \sqrt{1} = 1$$

$\tan\phi = \frac{\sqrt{3}/2}{1/2} = \sqrt{3} \rightarrow \phi = 60^\circ$

Step 2: $\frac{1}{2}\sin\theta + \frac{\sqrt{3}}{2}\cos\theta = \sin(\theta + 60^\circ)$

Answer: $\sin(\theta + 60^\circ)$

(vi) 13sin − **84**cos

Step 1: $a = 13, b = 84$

$$r = \sqrt{13^2 + 84^2} = \sqrt{169 + 7056} = \sqrt{7225} = 85\tan\phi = \frac{84}{13}$$

Step 2: $13\sin\theta - 84\cos\theta = 85\sin(\theta - \phi)$, where $\tan\phi = \frac{84}{13}$

Answer: $85\sin(\theta - \phi)$, $\tan\phi = \frac{84}{13}$

Summary Table

13	Proven
14	Proven
15(i)	$25\sin(\theta+\phi),\ \tan\phi=\frac{7}{24}$
15(ii)	$13\sin(\theta-\phi),\ \tan\phi=\frac{5}{12}$
15(iii)	$\sqrt{2}\sin(\theta-45^\circ)$
15(iv)	$10\sin(\theta-\phi),\ \tan\phi=\frac{3}{4}$
15(v)	$\sin(\theta+60^\circ)$
15(vi)	$85\sin(\theta-\phi),\ \tan\phi=\frac{84}{13}$