EXERCISE 10.4 - PRODUCT-TO-SUM AND SUM-TO-PRODUCT

Complete Solutions

Question 1: Express products as sums or differences

Formulas Used:

- $2\sin A\cos B = \sin(A+B) + \sin(A-B)$
- $2\cos A\sin B = \sin(A+B) - \sin(A-B)$
- $2\cos A\cos B = \cos(A+B) + \cos(A-B)$
- $2\sin A\sin B = \cos(A-B) - \cos(A+B)$
- $\sin A\cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$

(i) $\mathbf{2}\text{sin}\mathbf{3}$ cos

Step 1: Use formula: $2\sin A\cos B = \sin(A+B) + \sin(A-B)$

$$= \sin(3\theta+\theta) + \sin(3\theta-\theta) = \sin 4\theta + \sin 2\theta$$

Answer: $\sin 4\theta + \sin 2\theta$

(ii) $\mathbf{2}\text{cos}\mathbf{5}$ $\text{sin}\mathbf{3}$

Step 1: $2\cos A\sin B = \sin(A+B) - \sin(A-B)$

$$= \sin(5\theta+3\theta) - \sin(5\theta-3\theta) = \sin 8\theta - \sin 2\theta$$

Answer: $\sin 8\theta - \sin 2\theta$

(iii) $\text{sin}\mathbf{5}$ $\text{cos}\mathbf{2}$

Step 1: $\sin A\cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$

$$= \frac{1}{2}[\sin 7\theta + \sin 3\theta]$$

Answer: $\frac{1}{2}(\sin 7\theta + \sin 3\theta)$

(iv) $\mathbf{2}\text{sin}\mathbf{7}$ $\text{sin}\mathbf{2}$

Step 1: $2\sin A\sin B = \cos(A-B) - \cos(A+B)$

$$= \cos(7\theta-2\theta) - \cos(7\theta+2\theta) = \cos 5\theta - \cos 9\theta$$

Answer: $\cos 5\theta - \cos 9\theta$

(v) $\cos(\mathbf{x}+\mathbf{y})\sin(\mathbf{x}-\mathbf{y})$

Step 1: $\cos A\sin B = \frac{1}{2}[\sin(A+B) - \sin(A-B)]$

$$= \frac{1}{2}[\sin((x+y)+(x-y)) - \sin((x+y)-(x-y))] = \frac{1}{2}[\sin 2x - \sin 2y]$$

Answer: $\frac{1}{2}(\sin 2x - \sin 2y)$

(vi) $\cos(\mathbf{2x}+\mathbf{30}^{\circ})\cos(\mathbf{2x}-\mathbf{30}^{\circ})$

Step 1: $\cos A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$

$$= \frac{1}{2}[\cos(4x) + \cos(60^{\circ})] = \frac{1}{2}\cos 4x + \frac{1}{2}\cos 60^{\circ} = \frac{1}{2}\cos 4x + \frac{1}{4}$$

Answer: $\frac{1}{2}\cos 4x + \frac{1}{4}$

(vii) $\sin\mathbf{12}^{\circ}\sin\mathbf{46}^{\circ}$

Step 1: $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$

$$= \frac{1}{2}[\cos(34^{\circ}) - \cos(58^{\circ})]$$

Answer: $\frac{1}{2}(\cos 34^{\circ} - \cos 58^{\circ})$

(viii) $\sin(\mathbf{x}+\mathbf{45}^{\circ})\sin(\mathbf{x}-\mathbf{45}^{\circ})$

Step 1: $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$

$$= \frac{1}{2}[\cos(90^{\circ}) - \cos(2x)] = \frac{1}{2}[0 - \cos 2x] = -\frac{1}{2}\cos 2x$$

Answer: $-\frac{1}{2}\cos 2x$

Question 2: Express sums or differences as products

Formulas Used:

- $\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$
- $\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$
- $\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$
- $\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$

(i) $\sin\mathbf{5} + \sin\mathbf{3}$

Step 1: $\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$

$$= 2\sin\frac{8\theta}{2}\cos\frac{2\theta}{2} = 2\sin 4\theta\cos\theta$$

Answer: $2\sin 4\theta\cos\theta$

(ii) $\sin\mathbf{8} - \sin\mathbf{4}$

Step 1: $\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$

$$= 2\cos\frac{12\theta}{2}\sin\frac{4\theta}{2} = 2\cos 6\theta\sin 2\theta$$

Answer: $2\cos 6\theta\sin 2\theta$

(iii) $\cos\mathbf{6} + \cos\mathbf{3}$

Step 1: $\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$

$$= 2\cos\frac{9\theta}{2}\cos\frac{3\theta}{2}$$

Answer: $2\cos\frac{9\theta}{2}\cos\frac{3\theta}{2}$

(iv) $\cos\mathbf{7} - \cos$

Step 1: $\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$

$$= -2\sin\frac{8\theta}{2}\sin\frac{6\theta}{2} = -2\sin 4\theta\sin 3\theta$$

Answer: $-2\sin 4\theta\sin 3\theta$

(v) $\cos\mathbf{12}^\circ + \cos\mathbf{48}^\circ$

Step 1: $\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$

$$= 2\cos\frac{60^\circ}{2}\cos\frac{-36^\circ}{2} = 2\cos 30^\circ\cos 18^\circ = 2\cdot\frac{\sqrt{3}}{2}\cos 18^\circ = \sqrt{3}\cos 18^\circ$$

Answer: $\sqrt{3}\cos 18^\circ$

(vi) $\sin(\mathbf{x}+\mathbf{30}^\circ) + \sin(\mathbf{x}-\mathbf{30}^\circ)$

Step 1: $\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$

$$= 2\sin\frac{2x}{2}\cos\frac{60^\circ}{2} = 2\sin x\cos 30^\circ = \sqrt{3}\sin x$$

Answer: $\sqrt{3}\sin x$

Question 3: Prove the following identities

(i) $\frac{\sin\mathbf{3x} - \sin\mathbf{x}}{\cos\mathbf{x} - \cos\mathbf{3x}} = \cot\mathbf{2x}$

Step 1: Numerator: $\sin 3x - \sin x = 2\cos 2x\sin x$
Step 2: Denominator: $\cos x - \cos 3x = -2\sin 2x\sin(-x) = 2\sin 2x\sin x$

Step 3: LHS $= \frac{2\cos 2x\sin x}{2\sin 2x\sin x} = \frac{\cos 2x}{\sin 2x} = \cot 2x$

(ii) $\frac{\sin\mathbf{8x} + \sin\mathbf{2x}}{\cos\mathbf{8x} + \cos\mathbf{2x}} = \tan\mathbf{5x}$

Step 1: Numerator $= 2\sin 5x\cos 3x$
Step 2: Denominator $= 2\cos 5x\cos 3x$

Step 3: LHS $= \frac{2\sin 5x\cos 3x}{2\cos 5x\cos 3x} = \tan 5x$

(iii) $\frac{\sin\mathbf{A} - \sin\mathbf{B}}{\sin\mathbf{A} + \sin\mathbf{B}} = \tan\frac{\mathbf{A}-\mathbf{B}}{\mathbf{2}}\cot\frac{\mathbf{A}+\mathbf{B}}{\mathbf{2}}$

Step 1: Numerator $= 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$
Step 2: Denominator $= 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$

Step 3: LHS $= \frac{\cos\frac{A+B}{2}\sin\frac{A-B}{2}}{\sin\frac{A+B}{2}\cos\frac{A-B}{2}} = \tan\frac{A-B}{2}\cot\frac{A+B}{2}$

(iv) $\frac{\sin\mathbf{80}^{\circ}+\sin\mathbf{40}^{\circ}}{\cos\mathbf{80}^{\circ}+\cos\mathbf{40}^{\circ}}=\sqrt{\mathbf{3}}$

Step 1: Numerator $= 2\sin 60^{\circ}\cos 20^{\circ} = 2\cdot\frac{\sqrt{3}}{2}\cos 20^{\circ} = \sqrt{3}\cos 20^{\circ}$
Step 2: Denominator $= 2\cos 60^{\circ}\cos 20^{\circ} = 2\cdot\frac{1}{2}\cos 20^{\circ} = \cos 20^{\circ}$

Step 3: LHS $= \frac{\sqrt{3}\cos 20^{\circ}}{\cos 20^{\circ}} = \sqrt{3}$

Question 4: Prove the following

(i) $\cos\mathbf{15}^{\circ}+\cos\mathbf{105}^{\circ}+\cos\mathbf{195}^{\circ}+\cos\mathbf{165}^{\circ}+\cos\mathbf{285}^{\circ}=\mathbf{0}$

Step 1: Pair terms:

- $\cos 15^{\circ} + \cos 165^{\circ} = 2\cos 90^{\circ}\cos(-75^{\circ}) = 0$
- $\cos 105^{\circ} + \cos 195^{\circ} = 2\cos 150^{\circ}\cos(-45^{\circ}) = 2(-\frac{\sqrt{3}}{2})(\frac{\sqrt{2}}{2}) = -\frac{\sqrt{6}}{2}$

Step 2: $\cos 285^{\circ} = \cos(360^{\circ} - 75^{\circ}) = \cos 75^{\circ}$

Step 3: The sum $= 0 + 0 + \cos 75^{\circ} + \cos 105^{\circ} + \cos 195^{\circ}$

Step 4: $\cos 75^{\circ} + \cos 105^{\circ} = 2\cos 90^{\circ}\cos 15^{\circ} = 0$

Step 5: So total $= \cos 195^{\circ} = \cos(180^{\circ} + 15^{\circ}) = -\cos 15^{\circ}$

This doesn't give 0. Let's re-evaluate:

Actually, $\cos 105^{\circ} + \cos 195^{\circ} = 2\cos 150^{\circ}\cos(-45^{\circ}) = 2(-\frac{\sqrt{3}}{2})(\frac{\sqrt{2}}{2}) = -\frac{\sqrt{6}}{2}$

And $\cos 285^{\circ} = \cos(270^{\circ} + 15^{\circ}) = \sin 15^{\circ}$ — This doesn't cancel.

The identity may have a typo.

(ii) $\frac{\sin\mathbf{2}+\sin\mathbf{4}+\sin\mathbf{6}+\sin\mathbf{8}}{\cos\mathbf{2}+\cos\mathbf{4}+\cos\mathbf{6}+\cos\mathbf{8}}=\tan\mathbf{5}$

Step 1: Group numerator:

$$(\sin 2\theta + \sin 8\theta) + (\sin 4\theta + \sin 6\theta) = 2\sin 5\theta\cos 3\theta + 2\sin 5\theta\cos\theta = 2\sin 5\theta(\cos 3\theta + \cos\theta)$$

Step 2: Group denominator:

$$(\cos 2\theta + \cos 8\theta) + (\cos 4\theta + \cos 6\theta) = 2\cos 5\theta\cos 3\theta + 2\cos 5\theta\cos\theta = 2\cos 5\theta(\cos 3\theta + \cos\theta)$$

Step 3: LHS $= \frac{2\sin 5\theta(\cos 3\theta+\cos\theta)}{2\cos 5\theta(\cos 3\theta+\cos\theta)} = \tan 5\theta$

Summary Table

1(i)	$\sin 4\theta + \sin 2\theta$
1(ii)	$\sin 8\theta - \sin 2\theta$
1(iii)	$\frac{1}{2}(\sin 7\theta + \sin 3\theta)$
1(iv)	$\cos 5\theta - \cos 9\theta$
1(v)	$\frac{1}{2}(\sin 2x - \sin 2y)$
1(vi)	$\frac{1}{2}\cos 4x + \frac{1}{4}$
1(vii)	$\frac{1}{2}(\cos 34^{\circ} - \cos 58^{\circ})$
1(viii)	$-\frac{1}{2}\cos 2x$
2(i)	$2\sin 4\theta\cos\theta$
2(ii)	$2\cos 6\theta\sin 2\theta$

2(iii)	$2\cos\frac{9\theta}{2}\cos\frac{3\theta}{2}$
2(iv)	$-2\sin 4\theta\sin 3\theta$
2(v)	$\sqrt{3}\cos 18^\circ$
2(vi)	$\sqrt{3}\sin x$
3(i)-(iv)	All proven
4(ii)	Proven

EXERCISE 10.4 - PRODUCT-TO-SUM AND SUM-TO-PRODUCT (Continued)

(iii) $\cos^{\mathbf{2}}\left(\frac{}{\mathbf{4}}-\frac{}{\mathbf{2}}\right)-\cos^{\mathbf{2}}\left(\frac{}{\mathbf{4}}+\frac{}{\mathbf{2}}\right)=\sin$

Step 1: Use $\cos^2 A-\cos^2 B=-\sin(A+B)\sin(A-B)$

Step 2: Let $A=\frac{\pi}{4}-\frac{\alpha}{2}$, $B=\frac{\pi}{4}+\frac{\alpha}{2}$

- $A+B=\frac{\pi}{2}$
- $A-B=-\alpha$

Step 3: LHS $=-\sin\frac{\pi}{2}\sin(-\alpha)=-1\cdot(-\sin\alpha)=\sin\alpha$

(iv) $\sin\left(\frac{}{\mathbf{4}}-\ \right)\sin\left(\frac{}{\mathbf{4}}+\ \right)=\frac{\mathbf{1}}{\mathbf{2}}\cos\mathbf{2}$

Step 1: Use $\sin A\sin B=\frac{1}{2}[\cos(A-B)-\cos(A+B)]$

- $A=\frac{\pi}{4}-\theta$, $B=\frac{\pi}{4}+\theta$
- $A-B=-2\theta$, $A+B=\frac{\pi}{2}$

Step 2: LHS $=\frac{1}{2}[\cos(-2\theta)-\cos\frac{\pi}{2}]=\frac{1}{2}[\cos 2\theta-0]=\frac{1}{2}\cos 2\theta$

(v) $\frac{\sin\ +\sin\mathbf{3}\ +\sin\mathbf{5}\ +\sin\mathbf{7}}{\cos\ +\cos\mathbf{3}\ +\cos\mathbf{5}\ +\cos\mathbf{7}}=\tan\mathbf{4}$

Step 1: Numerator $=(\sin\theta+\sin 7\theta)+(\sin 3\theta+\sin 5\theta)$

$$=2\sin 4\theta\cos 3\theta+2\sin 4\theta\cos\theta=2\sin 4\theta(\cos 3\theta+\cos\theta)$$

Step 2: Denominator $=(\cos\theta+\cos 7\theta)+(\cos 3\theta+\cos 5\theta)$

$$=2\cos 4\theta\cos 3\theta+2\cos 4\theta\cos\theta=2\cos 4\theta(\cos 3\theta+\cos\theta)$$

Step 3: LHS $=\frac{2\sin 4\theta(\cos 3\theta+\cos\theta)}{2\cos 4\theta(\cos 3\theta+\cos\theta)}=\tan 4\theta$

Question 5: Prove the following

(i) $\cos\mathbf{20}^\circ\cos\mathbf{40}^\circ\cos\mathbf{60}^\circ\cos\mathbf{80}^\circ=\frac{\mathbf{1}}{\mathbf{16}}$

Step 1: $\cos 60^\circ=\frac{1}{2}$

Step 2: $\cos 20^\circ\cos 40^\circ\cos 80^\circ=\frac{1}{4}\cos 60^\circ=\frac{1}{8}$

Step 3: Wait, let's use the identity: $\cos A\cos(60^\circ-A)\cos(60^\circ+A)=\frac{1}{4}\cos 3A$

For $A=20^\circ$: $\cos 20^\circ\cos 40^\circ\cos 80^\circ=\frac{1}{4}\cos 60^\circ=\frac{1}{4}\cdot\frac{1}{2}=\frac{1}{8}$

Step 4: Multiply by $\cos 60^\circ=\frac{1}{2}$:

$\cos 20^\circ \cos 40^\circ \cos 60^\circ \cos 80^\circ = \frac{1}{8} \cdot \frac{1}{2} = \frac{1}{16}$

(ii) $\sin\frac{}{\mathbf{9}}\sin\frac{\mathbf{2}}{\mathbf{9}}\sin\frac{\mathbf{3}}{\mathbf{9}}\sin\frac{\mathbf{4}}{\mathbf{9}}=\frac{\mathbf{3}}{\mathbf{16}}$

Step 1: $\sin\frac{3\pi}{9} = \sin 60^\circ = \frac{\sqrt{3}}{2}$

Step 2: $\sin\frac{\pi}{9}\sin\frac{2\pi}{9}\sin\frac{3\pi}{9}\sin\frac{4\pi}{9} = \frac{\sqrt{3}}{2}\sin 20^\circ \sin 40^\circ \sin 80^\circ$

Step 3: $\sin 20^\circ \sin 40^\circ \sin 80^\circ = \frac{\sqrt{3}}{8}$

Step 4: Product $= \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{8} = \frac{3}{16}$

(iii) $\sin\mathbf{10}^\circ\sin\mathbf{30}^\circ\sin\mathbf{50}^\circ\sin\mathbf{70}^\circ=\frac{\mathbf{1}}{\mathbf{16}}$

Step 1: $\sin 30^\circ = \frac{1}{2}$

Step 2: $\sin 10^\circ \sin 50^\circ \sin 70^\circ = \sin 10^\circ \cos 20^\circ \cos 40^\circ$

Step 3: Using $\sin 10^\circ \cos 20^\circ \cos 40^\circ = \frac{1}{8}$

Step 4: Product $= \frac{1}{2} \cdot \frac{1}{8} = \frac{1}{16}$

Question 6: Prove $\sin\mathbf{3}\ =\sin$ **deduce** $\sin\mathbf{15}^\circ$

Step 1: $\sin 3\theta = \sin\theta$

$$3\sin\theta - 4\sin^3\theta = \sin\theta 2\sin\theta - 4\sin^3\theta = 0 2\sin\theta(1 - 2\sin^2\theta) = 0$$

Step 2: For $\theta \neq 0$: $1 - 2\sin^2\theta = 0 \to \sin^2\theta = \frac{1}{2} \to \sin\theta = \frac{1}{\sqrt{2}}$

Step 3: For $\theta = 15^\circ$, $\sin 45^\circ = \sin 15^\circ$ gives $\sin 15^\circ = \frac{1}{\sqrt{2}}$

But actually $\sin 45^\circ = \frac{1}{\sqrt{2}}$ is correct for $\theta = 15^\circ$ since $\sin 45^\circ = \sin 15^\circ$ is not true.

The problem statement is ambiguous.

Question 7: $\tan\mathbf{75}^\circ-\tan\mathbf{15}^\circ=\mathbf{2}\sqrt{\mathbf{3}}$

Step 1: $\tan 75^\circ = \cot 15^\circ = \frac{1}{\tan 15^\circ}$

Step 2: $\tan 15^\circ = 2 - \sqrt{3}$

Step 3: $\tan 75^\circ = \frac{1}{2-\sqrt{3}} = 2 + \sqrt{3}$

Step 4: $\tan 75^\circ - \tan 15^\circ = (2 + \sqrt{3}) - (2 - \sqrt{3}) = 2\sqrt{3}$

Question 8: $\cos\mathbf{15}^\circ-\sin\mathbf{15}^\circ=\frac{\mathbf{1}}{\sqrt{\mathbf{2}}}$

Step 1: $\cos 15^\circ = \frac{\sqrt{6}+\sqrt{2}}{4}$, $\sin 15^\circ = \frac{\sqrt{6}-\sqrt{2}}{4}$

Step 2: $\cos 15^\circ - \sin 15^\circ = \frac{(\sqrt{6}+\sqrt{2})-(\sqrt{6}-\sqrt{2})}{4} = \frac{2\sqrt{2}}{4} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$

Question 9: $\frac{\sin^{\mathbf{2}}\ -\sin^{\mathbf{2}}}{\sin\ \cos\ -\sin\ \cos}=\tan(\ +\ )$

Step 1: Numerator $= \sin(\alpha+\beta)\sin(\alpha-\beta)$

Step 2: Denominator $= \frac{1}{2}(\sin 2\alpha - \sin 2\beta) = \frac{1}{2} \cdot 2\cos(\alpha+\beta)\sin(\alpha-\beta) = \cos(\alpha+\beta)\sin(\alpha-\beta)$

Step 3: LHS $= \frac{\sin(\alpha+\beta)\sin(\alpha-\beta)}{\cos(\alpha+\beta)\sin(\alpha-\beta)} = \tan(\alpha+\beta)$

Question 10: $\sin\ +\sin\ +\sin\ -\sin(\ +\ +\ ) = \mathbf{4}\sin\frac{+}{\mathbf{2}}\sin\frac{+}{\mathbf{2}}\sin\frac{+}{\mathbf{2}}$

Step 1: $\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$

Step 2: $\sin\gamma - \sin(\alpha+\beta+\gamma) = -2\cos\frac{\alpha+\beta+2\gamma}{2}\sin\frac{\alpha+\beta}{2}$

Step 3: LHS $= 2\sin\frac{\alpha+\beta}{2}\left[\cos\frac{\alpha-\beta}{2} - \cos\frac{\alpha+\beta+2\gamma}{2}\right]$

Step 4: Use $\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$

- $A = \frac{\alpha-\beta}{2}$, $B = \frac{\alpha+\beta+2\gamma}{2}$
- $\frac{A+B}{2} = \frac{\alpha+\gamma}{2}$, $\frac{A-B}{2} = -\frac{\beta+\gamma}{2}$

Step 5: LHS $= 2\sin\frac{\alpha+\beta}{2}\cdot\left[-2\sin\frac{\alpha+\gamma}{2}\sin\left(-\frac{\beta+\gamma}{2}\right)\right]$

$= 4\sin\frac{\alpha+\beta}{2}\sin\frac{\beta+\gamma}{2}\sin\frac{\gamma+\alpha}{2}$

Summary Table

4(iii)	Proven
4(iv)	Proven
4(v)	Proven
5(i)	Proven
5(ii)	Proven
5(iii)	Proven
7	Proven
8	Proven
9	Proven
10	Proven