EXERCISE 12.1 - LIMITS OF SEQUENCES

Complete Solutions

Question 1: Find the limit of the following sequences if exists

(i) $\mathbf{a_n = \frac{2n+3}{n+1}}$

Step 1: Divide numerator and denominator by n (highest power in denominator)

$$a_n = \frac{2n+3}{n+1} = \frac{2+\frac{3}{n}}{1+\frac{1}{n}}$$

Step 2: As $n \to \infty$, $\frac{3}{n} \to 0$ and $\frac{1}{n} \to 0$

Step 3: $\lim_{n\to\infty} a_n = \frac{2+0}{1+0} = 2$

Answer: 2

(ii) $\mathbf{b_n = \frac{2n+3}{n^2+1}}$

Step 1: Divide numerator and denominator by n^2 (highest power in denominator)

$$b_n = \frac{2n+3}{n^2+1} = \frac{\frac{2}{n}+\frac{3}{n^2}}{1+\frac{1}{n^2}}$$

Step 2: As $n \to \infty$, $\frac{2}{n} \to 0$, $\frac{3}{n^2} \to 0$, and $\frac{1}{n^2} \to 0$

Step 3: $\lim_{n\to\infty} b_n = \frac{0+0}{1+0} = 0$

Answer: 0

(iii) $\mathbf{c_n = \frac{5n^2}{2n+3}}$

Step 1: Divide numerator and denominator by n (highest power in denominator)

$$c_n = \frac{5n^2}{2n+3} = \frac{5n}{2+\frac{3}{n}}$$

Step 2: As $n \to \infty$, numerator $5n \to \infty$, denominator $2+\frac{3}{n} \to 2$

Step 3: $\lim_{n\to\infty} c_n = \infty$ (diverges)

Answer: Does not exist (diverges to infinity)

(iv) $\mathbf{d_n = \frac{n^2-3n+1}{2n^2+n+4}}$

Step 1: Divide numerator and denominator by n^2 (highest power overall)

$$d_n = \frac{n^2-3n+1}{2n^2+n+4} = \frac{1-\frac{3}{n}+\frac{1}{n^2}}{2+\frac{1}{n}+\frac{4}{n^2}}$$

Step 2: As $n \to \infty$, all terms with $\frac{1}{n}$ or $\frac{1}{n^2}$ tend to 0

Step 3: $\lim_{n\to\infty} d_n = \frac{1-0+0}{2+0+0} = \frac{1}{2}$

Answer: $\frac{1}{2}$

Summary Table

Sequence	Limit
$a_n = \frac{2n+3}{n+1}$	2
$b_n = \frac{2n+3}{n^2+1}$	0
$c_n = \frac{5n^2}{2n+3}$	Does not exist (diverges)
$d_n = \frac{n^2-3n+1}{2n^2+n+4}$	$\frac{1}{2}$

EXERCISE 12.1 - LIMITS (Continued)

Complete Solutions

Question 2: Evaluate each limit by using theorems of limits

(i) $\lim_{x\to 3}(2x+4)$

Step 1: Direct substitution:

$$\lim_{x\to 3}(2x+4) = 2(3)+4 = 10$$

Answer: 10

(ii) $\lim_{x\to 1}(3x^2-2x+4)$

Step 1: Direct substitution:

$$3(1)^2-2(1)+4 = 3-2+4 = 5$$

Answer: 5

(iii) $\lim_{x\to 3}\sqrt{x^2+x+4}$

Step 1: Direct substitution:

$$\sqrt{(3)^2+3+4} = \sqrt{9+3+4} = \sqrt{16} = 4$$

Answer: 4

(iv) $\lim_{x\to 2}\sqrt{x^2+4}$

Step 1: Direct substitution:

$$\sqrt{2^2+4} = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$$

Answer: $2\sqrt{2}$

(v) $\lim_{x\to 2}\sqrt{x^2+1}-\sqrt{x^2+5}$

Step 1: Direct substitution:

$$\sqrt{4+1}-\sqrt{4+5} = \sqrt{5}-\sqrt{9} = \sqrt{5}-3$$

Answer: $\sqrt{5}-3$

(vi) $\lim_{\mathbf{x}\to 2} \frac{\mathbf{2x^3+5x}}{\mathbf{3x-2}}$

Step 1: Direct substitution:

$$\frac{2(2)^3+5(2)}{3(2)-2} = \frac{16+10}{6-2} = \frac{26}{4} = \frac{13}{2}$$

Answer: $\frac{13}{2}$

Question 3: Evaluate each limit by using algebraic techniques

(i) $\lim_{\mathbf{x}\to 1} \frac{\mathbf{x^3-x}}{\mathbf{x+1}}$

Step 1: Factor numerator: $x^3 - x = x(x^2-1) = x(x-1)(x+1)$

Step 2: Cancel $(x+1)$:

$$\lim_{x\to 1} \frac{x(x-1)(x+1)}{x+1} = \lim_{x\to 1} x(x-1) = 1(0) = 0$$

Answer: 0

(ii) $\lim_{\mathbf{x}\to 3} \frac{\mathbf{x^2-5x+6}}{\mathbf{x^2-2x-3}}$

Step 1: Factor numerator: $x^2-5x+6 = (x-2)(x-3)$
Step 2: Factor denominator: $x^2-2x-3 = (x-3)(x+1)$

Step 3: Cancel $(x-3)$:

$$\lim_{x\to 3} \frac{(x-2)(x-3)}{(x-3)(x+1)} = \lim_{x\to 3} \frac{x-2}{x+1} = \frac{3-2}{3+1} = \frac{1}{4}$$

Answer: $\frac{1}{4}$

(iii) $\lim_{\mathbf{x}\to 2} \frac{\mathbf{x^3-8}}{\mathbf{x^2-5x+6}}$

Step 1: Factor numerator: $x^3-8 = (x-2)(x^2+2x+4)$
Step 2: Factor denominator: $x^2-5x+6 = (x-2)(x-3)$

Step 3: Cancel $(x-2)$:

$$\lim_{x\to 2} \frac{(x-2)(x^2+2x+4)}{(x-2)(x-3)} = \lim_{x\to 2} \frac{x^2+2x+4}{x-3} = \frac{4+4+4}{2-3} = \frac{12}{-1} = -12$$

Answer: -12

(iv) $\lim_{\mathbf{x}\to 1} \frac{\mathbf{x^3-3x^2+3x-1}}{\mathbf{x^2-x}}$ **(Note: Numerator is $(\mathbf{x-1})^3$)**

Step 1: Factor numerator: $x^3-3x^2+3x-1 = (x-1)^3$
Step 2: Factor denominator: $x^2-x = x(x-1)$

Step 3: Cancel $(x-1)$:

$$\lim_{x\to 1} \frac{(x-1)^3}{x(x-1)} = \lim_{x\to 1} \frac{(x-1)^2}{x} = \frac{0}{1} = 0$$

Answer: 0

(v) $\lim_{\mathbf{x\to 2}} \frac{\mathbf{x^3-6x^2+12x-8}}{\mathbf{x^3-4x}}$

Step 1: Factor numerator: $x^3 - 6x^2 + 12x - 8 = (x-2)^3$
Step 2: Factor denominator: $x^3 - 4x = x(x^2-4) = x(x-2)(x+2)$

Step 3: Cancel $(x-2)$:

$$\lim_{x\to 2} \frac{(x-2)^3}{x(x-2)(x+2)} = \lim_{x\to 2} \frac{(x-2)^2}{x(x+2)} = \frac{0}{2(4)} = 0$$

Answer: 0

(vi) $\lim_{\mathbf{x\to 1}} \frac{\mathbf{x^4-1}}{\mathbf{x^3-3x+2}}$

Step 1: Factor numerator: $x^4 - 1 = (x-1)(x+1)(x^2+1)$
Step 2: Factor denominator: $x^3 - 3x + 2$

- $x = 1$ is a root: $1 - 3 + 2 = 0$
- Divide: $x^3 - 3x + 2 = (x-1)(x^2+x-2) = (x-1)(x+2)(x-1) = (x-1)^2(x+2)$

Step 3: Cancel $(x-1)$:

$$\lim_{x\to 1} \frac{(x-1)(x+1)(x^2+1)}{(x-1)^2(x+2)} = \lim_{x\to 1} \frac{(x+1)(x^2+1)}{(x-1)(x+2)}$$

This gives $\frac{2\cdot 2}{0} = \infty$ (still undefined)

Step 4: Let's re-evaluate: The limit does not exist (infinite).

Answer: Does not exist (diverges to infinity)

(vii) $\lim_{\mathbf{x\to 2}} \frac{\mathbf{x-2}}{\sqrt{\mathbf{x+2}}-\sqrt{\mathbf{6-x}}}$

Step 1: Rationalize the denominator:

$$\frac{x-2}{\sqrt{x+2}-\sqrt{6-x}} \times \frac{\sqrt{x+2}+\sqrt{6-x}}{\sqrt{x+2}+\sqrt{6-x}} = \frac{(x-2)(\sqrt{x+2}+\sqrt{6-x})}{(x+2)-(6-x)} = \frac{(x-2)(\sqrt{x+2}+\sqrt{6-x})}{2x-4}$$

Step 2: $2x - 4 = 2(x-2)$

Step 3: Cancel $(x-2)$:

$$\lim_{x\to 2} \frac{\sqrt{x+2}+\sqrt{6-x}}{2} = \frac{\sqrt{4}+\sqrt{4}}{2} = \frac{2+2}{2} = 2$$

Answer: 2

(viii) $\lim_{\mathbf{h\to 0}} \frac{\sqrt{\mathbf{x+h}}-\sqrt{\mathbf{x}}}{\mathbf{h}}$

Step 1: Rationalize the numerator:

$$\frac{\sqrt{x+h}-\sqrt{x}}{h} \times \frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}} = \frac{(x+h)-x}{h(\sqrt{x+h}+\sqrt{x})} = \frac{h}{h(\sqrt{x+h}+\sqrt{x})} = \frac{1}{\sqrt{x+h}+\sqrt{x}}$$

Step 2: $\lim_{h\to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}} = \frac{1}{2\sqrt{x}}$

Answer: $\frac{1}{2\sqrt{x}}$

(ix) $\lim_{\mathbf{x}\to\mathbf{0}}\frac{\mathbf{x^2-a^2}}{\mathbf{x^2-a^2}}$

Step 1: Direct substitution: $\frac{0-a^2}{0-a^2}=1$

Answer: 1

Question 4: Evaluate the following limits

(i) $\lim_{\mathbf{x}\to\mathbf{0}}\frac{\sin\mathbf{5x}}{\mathbf{x}}$

Step 1: Use $\lim_{x\to 0}\frac{\sin ax}{x}=a$

$$\lim_{x\to 0}\frac{\sin 5x}{x}=5$$

Answer: 5

(ii) $\lim_{\mathbf{x}\to\mathbf{0}}\frac{\sin\mathbf{x^2}}{\mathbf{x}}$

Step 1: $\frac{\sin x^2}{x}=\frac{\sin x^2}{x^2}\cdot x$

$$\lim_{x\to 0}\frac{\sin x^2}{x^2}\cdot\lim_{x\to 0}x=1\cdot 0=0$$

Answer: 0

(iii) $\lim_{\to\mathbf{0}}\frac{\mathbf{1}-\cos}{\sin}$

Step 1: $1-\cos\theta=2\sin^2\frac{\theta}{2}$, $\sin\theta=2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$

Step 2: $\frac{1-\cos\theta}{\sin\theta}=\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}=\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}=\tan\frac{\theta}{2}$

Step 3: $\lim_{\theta\to 0}\tan\frac{\theta}{2}=0$

Answer: 0

(iv) $\lim_{\mathbf{x}\to\frac{}{\mathbf{4}}}\frac{\sin\mathbf{x}-\cos\mathbf{x}}{\mathbf{x}-\frac{}{\mathbf{4}}}$

Step 1: Let $h=x-\frac{\pi}{4}$, so $x=\frac{\pi}{4}+h$

Step 2: $\sin(\frac{\pi}{4}+h)-\cos(\frac{\pi}{4}+h)$

$$=\left(\frac{1}{\sqrt{2}}\cos h+\frac{1}{\sqrt{2}}\sin h\right)-\left(\frac{1}{\sqrt{2}}\cos h-\frac{1}{\sqrt{2}}\sin h\right)=\frac{2}{\sqrt{2}}\sin h=\sqrt{2}\sin h$$

Step 3: $\lim_{h\to 0}\frac{\sqrt{2}\sin h}{h}=\sqrt{2}$

Answer: $\sqrt{2}$

(v) $\lim_{\mathbf{x}\to\mathbf{0}}\frac{\cos\mathbf{ax}-\cos\mathbf{bx}}{\mathbf{x^2}}$

Step 1: Use $\cos A-\cos B=-2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$

$$\cos ax-\cos bx=-2\sin\frac{(a+b)x}{2}\sin\frac{(a-b)x}{2}$$

Step 2: $\frac{\cos ax - \cos bx}{x^2} = -2 \cdot \frac{\sin \frac{(a+b)x}{2}}{x} \cdot \frac{\sin \frac{(a-b)x}{2}}{x}$

$$= -2 \cdot \frac{a+b}{2} \cdot \frac{a-b}{2} \cdot \frac{\sin \frac{(a+b)x}{2}}{\frac{(a+b)x}{2}} \cdot \frac{\sin \frac{(a-b)x}{2}}{\frac{(a-b)x}{2}}$$

Step 3: As $x \to 0$, the sine ratios tend to 1:

$$= -2 \cdot \frac{a+b}{2} \cdot \frac{a-b}{2} = -\frac{(a+b)(a-b)}{2} = \frac{b^2 - a^2}{2}$$

Answer: $\frac{b^2-a^2}{2}$

(vi) $\lim_{\mathbf{x} \to \frac{\ }{4}} \frac{\tan \mathbf{x} - \mathbf{1}}{\mathbf{x} - \frac{\ }{4}}$

Step 1: Let $h = x - \frac{\pi}{4}$, so $x = \frac{\pi}{4} + h$

Step 2: $\tan(\frac{\pi}{4} + h) - 1 = \frac{1+\tan h}{1-\tan h} - 1 = \frac{2\tan h}{1-\tan h}$

Step 3: $\lim_{h \to 0} \frac{\frac{2\tan h}{1-\tan h}}{h} = \lim_{h \to 0} \frac{2}{1-\tan h} \cdot \frac{\tan h}{h} = 2 \cdot 1 = 2$

Answer: 2

(vii) $\lim_{\mathbf{x} \to \mathbf{0}} \frac{\mathbf{1} - \cos \mathbf{2x}}{\mathbf{x^2}}$

Step 1: $1 - \cos 2x = 2\sin^2 x$

Step 2: $\frac{2\sin^2 x}{x^2} = 2\left(\frac{\sin x}{x}\right)^2$

Step 3: $\lim_{x \to 0} 2\left(\frac{\sin x}{x}\right)^2 = 2(1)^2 = 2$

Answer: 2

(viii) $\lim_{\mathbf{x} \to \mathbf{0}} \frac{\cos \mathbf{ax} - \cos \mathbf{bx}}{\cos \mathbf{cx} - \cos \mathbf{dx}}$

Step 1: Using $\cos A - \cos B = -2\sin\frac{A+B}{2} \sin\frac{A-B}{2}$:

$$\lim_{x \to 0} \frac{-2\sin\frac{(a+b)x}{2} \sin\frac{(a-b)x}{2}}{-2\sin\frac{(c+d)x}{2} \sin\frac{(c-d)x}{2}} = \lim_{x \to 0} \frac{\sin\frac{(a+b)x}{2} \sin\frac{(a-b)x}{2}}{\sin\frac{(c+d)x}{2} \sin\frac{(c-d)x}{2}}$$

Step 2: As $x \to 0$, $\frac{\sin(kx)}{kx} \to 1$:

$$= \frac{\frac{a+b}{2} \cdot \frac{a-b}{2}}{\frac{c+d}{2} \cdot \frac{c-d}{2}} = \frac{(a+b)(a-b)}{(c+d)(c-d)} = \frac{a^2 - b^2}{c^2 - d^2}$$

Answer: $\frac{a^2-b^2}{c^2-d^2}$

(ix) $\lim_{\mathbf{x} \to \mathbf{1}} \frac{\mathbf{x^3} - \mathbf{1}}{\mathbf{x^2} - \mathbf{1}}$

Step 1: Factor:

$$\frac{x^3 - 1}{x^2 - 1} = \frac{(x-1)(x^2+x+1)}{(x-1)(x+1)} = \frac{x^2+x+1}{x+1}$$

Step 2: $\lim_{x \to 1} \frac{1+1+1}{1+1} = \frac{3}{2}$

Answer: $\frac{3}{2}$

(x) $\lim_{\mathbf{x \to 3}} \frac{\mathbf{x^2 - x\ln x + 3\ln x - 9}}{\mathbf{x - 3}}$

Step 1: Factor numerator: $x^2 - 9 - x\ln x + 3\ln x = (x-3)(x+3) - (x-3)\ln x$

Step 2: $= (x-3)(x+3-\ln x)$

Step 3: $\lim_{x\to 3} \frac{(x-3)(x+3-\ln x)}{x-3} = \lim_{x\to 3}(x+3-\ln x) = 6 - \ln 3$

Answer: $6 - \ln 3$

(xi) $\lim_{\mathbf{x\to 0}} \frac{\mathbf{x(2^x - 1)}}{\mathbf{1 - \cos x}}$

Step 1: $1 - \cos x = 2\sin^2 \frac{x}{2}$

Step 2: $\frac{x(2^x-1)}{1-\cos x} = \frac{x(2^x-1)}{2\sin^2\frac{x}{2}} = \frac{2^x-1}{x} \cdot \frac{x^2}{2\sin^2\frac{x}{2}} = \frac{2^x-1}{x} \cdot 2 \cdot \left(\frac{\frac{x}{2}}{\sin\frac{x}{2}}\right)^2$

Step 3: $\lim_{x\to 0} \frac{2^x-1}{x} = \ln 2$, and $\lim_{x\to 0} \left(\frac{\frac{x}{2}}{\sin\frac{x}{2}}\right)^2 = 1$

Step 4: Limit $= (\ln 2) \cdot 2 \cdot 1 = 2\ln 2$

Answer: $2\ln 2$

Question 5: Express each limit in terms of e

(i) $\lim_{\mathbf{n\to\infty}} \left(\mathbf{1 + \frac{1}{n}}\right)^{\mathbf{n}} = \mathbf{e}$

Answer: e

(ii) $\lim_{\mathbf{n\to\infty}} \left(\mathbf{1 + \frac{1}{n}}\right)^{\mathbf{n/2}}$

Step 1: $= \left[\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n\right]^{1/2} = e^{1/2} = \sqrt{e}$

Answer: $\sqrt{e}$

(iii) $\lim_{\mathbf{n\to\infty}} \left(\mathbf{1 - \frac{1}{n}}\right)^{\mathbf{n}}$

Step 1: $= \lim_{n\to\infty} \left(1 + \frac{-1}{n}\right)^n = e^{-1} = \frac{1}{e}$

Answer: $\frac{1}{e}$

(iv) $\lim_{\mathbf{n\to\infty}} \left(\mathbf{1 + \frac{1}{3n}}\right)^{\mathbf{n}}$

Step 1: Let $m = 3n$, then $n = \frac{m}{3}$

$$= \lim_{m\to\infty} \left(1 + \frac{1}{m}\right)^{m/3} = e^{1/3}$$

Answer: $e^{1/3}$

(v) $\lim_{\mathbf{n\to\infty}} \left(\mathbf{1 + \frac{4}{n}}\right)^{\mathbf{n}}$

Step 1: $= \lim_{n\to\infty} \left(1 + \frac{4}{n}\right)^n = e^4$

Answer: e^4

(vi) $\lim_{\mathbf{x\to\infty}} (\mathbf{1 + 3x})^{\mathbf{1/x}}$

Step 1: This is $\lim_{x\to\infty}(3x+1)^{1/x}$

Step 2: $= \lim_{x\to\infty} e^{\frac{1}{x}\ln(3x+1)} = e^{\lim_{x\to\infty}\frac{\ln(3x+1)}{x}} = e^0 = 1$

Answer: 1

(vii) $\lim_{x\to 0}(1+2x^2)^{1/x}$

Step 1: $= e^{\lim_{x\to 0}\frac{\ln(1+2x^2)}{x}} = e^{\lim_{x\to 0}\frac{2x^2}{x}} = e^{\lim_{x\to 0}2x} = e^0 = 1$

Answer: 1

(viii) $\lim_{x\to 0}\frac{e^x-e^{ax}}{abx}$

Step 1: $e^x - e^{ax} = e^{ax}(e^{(1-a)x}-1)$

Step 2: $\lim_{x\to 0}\frac{e^{ax}}{ab}\cdot\frac{e^{(1-a)x}-1}{x} = \frac{1}{ab}\cdot(1-a) = \frac{1-a}{ab}$

Answer: $\frac{1-a}{ab}$

(ix) $\lim_{x\to\infty}\left(\frac{x}{1+x}\right)^x$

Step 1: $\left(\frac{x}{1+x}\right)^x = \left(1-\frac{1}{1+x}\right)^x$

Step 2: Let $n = 1+x$, as $x\to\infty$, $n\to\infty$:

$$= \lim_{n\to\infty}\left(1-\frac{1}{n}\right)^{n-1} = \frac{1}{e}$$

Answer: $\frac{1}{e}$

(x) $\lim_{x\to 0}\frac{e^x-1}{x} = 1$

Answer: 1

(xi) $\lim_{x\to 0}\frac{e^x-1}{e^x+1} = \frac{0}{2} = 0$

Answer: 0

(xii) $\lim_{x\to 2}\frac{e^x-e^2}{x-2}$

Step 1: This is the derivative of e^x at $x=2$:

$$= e^2$$

Answer: e^2

Summary Table

2(i)	10
2(ii)	5
2(iii)	4
2(iv)	$2\sqrt{2}$
2(v)	$\sqrt{5}-3$
2(vi)	$\frac{13}{2}$
3(i)	0

3(ii)	$\frac{1}{4}$
3(iii)	-12
3(iv)	0
3(v)	0
3(vi)	Diverges
3(vii)	2
3(viii)	$\frac{1}{2\sqrt{x}}$
3(ix)	1
4(i)	5
4(ii)	0
4(iii)	0
4(iv)	$\sqrt{2}$
4(v)	$\frac{b^2-a^2}{2}$
4(vi)	2
4(vii)	2
4(viii)	$\frac{a^2-b^2}{c^2-d^2}$
4(ix)	$\frac{3}{2}$
4(x)	$6-\ln 3$
4(xi)	$2\ln 2$
5(i)	e
5(ii)	$\sqrt{e}$
5(iii)	$\frac{1}{e}$
5(iv)	$e^{1/3}$
5(v)	e^4
5(vi)	1
5(vii)	1
5(viii)	$\frac{1-a}{ab}$
5(ix)	$\frac{1}{e}$
5(x)	1
5(xi)	0
5(xii)	e^2