EXERCISE 12.3 - APPLICATIONS OF LIMITS

Complete Solutions

Question 1: Exponential Decay

Given: $\mathbf{A(t) = A_0 e^{-0.1t}}$, where $\mathbf{A_0}$ is the initial amount.

Step 1: Find $\lim_{t\to\infty} A(t)$:

$$\lim_{t\to\infty} A_0 e^{-0.1t}$$

Step 2: As $t \to \infty$, $-0.1t \to -\infty$, so $e^{-0.1t} \to 0$

Step 3: Therefore:

$$\lim_{t\to\infty} A(t) = A_0 \cdot 0 = 0$$

Answer: The substance decays to 0 as $t \to \infty$

Question 2: Population Growth Model (Logistic Growth)

Given: $\mathbf{P(t) = \frac{100{,}000}{1+9e^{-0.5t}}}$

Step 1: Find $\lim_{t\to\infty} P(t)$:

$$\lim_{t\to\infty} \frac{100,000}{1+9e^{-0.5t}}$$

Step 2: As $t \to \infty$, $-0.5t \to -\infty$, so $e^{-0.5t} \to 0$

Step 3: Denominator $\to 1 + 9(0) = 1$

Step 4: Therefore:

$$\lim_{t\to\infty} P(t) = \frac{100,000}{1} = 100,000$$

Answer: The long-term population is 100,000

Summary Table

1	0
2	100,000

EXERCISE 12.3 - APPLICATIONS OF LIMITS (Continued)

Question 3: Company Sales

Given: $\mathbf{S(t) = \frac{500t}{t+10}}$

Step 1: Find $\lim_{t\to\infty} S(t)$:

$$\lim_{t\to\infty} \frac{500t}{t+10}$$

Step 2: Divide numerator and denominator by t:

$$\lim_{t\to\infty} \frac{500}{1+\frac{10}{t}}$$

Step 3: As $t \to \infty$, $\frac{10}{t} \to 0$:

$$= \frac{500}{1+0} = 500$$

Interpretation: The weekly sales approach 500,000 (since S is in thousands) as time goes to infinity. This represents the maximum possible weekly sales (saturation point).

Answer: $\lim_{t\to\infty} S(t) = 500$. It represents the maximum sales capacity of 500,000 units.

Question 4: Signal Strength

Given: $\mathbf{S(d) = \frac{1000}{d^2}}$

(i) Signal strength at $\mathbf{d = 10}$

Step 1: Substitute $d = 10$:

$$S(10) = \frac{1000}{10^2} = \frac{1000}{100} = 10$$

Answer: 10 units

(ii) What happens as $\mathbf{d \to \infty}$?

Step 1: $\lim_{d\to\infty} \frac{1000}{d^2}$

Step 2: As $d \to \infty$, $d^2 \to \infty$, so $\frac{1000}{d^2} \to 0$

Answer: Signal strength approaches 0 as distance increases indefinitely.

Question 5: Stock Price Growth

Given: $\mathbf{P(t) = 50e^{0.05t}}$

(i) Limit as $\mathbf{t \to \infty}$

Step 1: $\lim_{t\to\infty} 50e^{0.05t}$

Step 2: As $t \to \infty$, $0.05t \to \infty$, so $e^{0.05t} \to \infty$

Answer: $\lim_{t\to\infty} P(t) = \infty$ (unbounded growth)

(ii) Price after 10 years

Step 1: $P(10) = 50e^{0.05(10)} = 50e^{0.5}$

Step 2: $e^{0.5} \approx 1.6487$

$$P(10) \approx 50(1.6487) = 82.435$$

Answer: Approximately 82.44

Question 6: Cost Function Continuity

Given:

$$C(x) = \begin{cases} 10x + 500 & \text{if } x \leq 100 \\ 12x + 300 & \text{if } x > 100 \end{cases}$$

Step 1: Function value at $x = 100$:

$$C(100) = 10(100) + 500 = 1000 + 500 = 1500$$

Step 2: Left hand limit ($x \to 100^-$):

$$\lim_{x \to 100^-} C(x) = \lim_{x \to 100^-} (10x + 500) = 1000 + 500 = 1500$$

Step 3: Right hand limit ($x \to 100^+$):

$$\lim_{x \to 100^+} C(x) = \lim_{x \to 100^+} (12x + 300) = 1200 + 300 = 1500$$

Step 4: LHL = RHL = C(100) = 1500

Answer: Yes, the cost function is continuous at $x = 100$.

Question 7: Inflation Model

Given: $\mathbf{I(t) = I_0 e^{0.05t}}$, $\mathbf{I_0 = 100}$

(i) Inflation rate after 5 years

Step 1: $I(5) = 100e^{0.05(5)} = 100e^{0.25}$

Step 2: $e^{0.25} \approx 1.2840$

$$I(5) \approx 100(1.2840) = 128.40$$

Answer: Price index after 5 years is approximately 128.40

(ii) Expected price index after 10 years

Step 1: $I(10) = 100e^{0.05(10)} = 100e^{0.5}$

Step 2: $e^{0.5} \approx 1.6487$

$$I(10) \approx 100(1.6487) = 164.87$$

Answer: Price index after 10 years is approximately 164.87

Question 8: Cost Function Continuity

Given:

$$C(x) = \begin{cases} 5x + 20 & \text{if } x \leq 10 \\ 6x + 10 & \text{if } x > 10 \end{cases}$$

Step 1: Function value at $x = 10$:

$$C(10) = 5(10) + 20 = 50 + 20 = 70$$

Step 2: Left hand limit ($x \to 10^-$):

$$\lim_{x\to 10^-} C(x) = \lim_{x\to 10^-} (5x + 20) = 50 + 20 = 70$$

Step 3: Right hand limit ($x \to 10^+$):

$$\lim_{x\to 10^+} C(x) = \lim_{x\to 10^+} (6x + 10) = 60 + 10 = 70$$

Step 4: LHL = RHL = C(10) = 70

Answer: Yes, the cost function is continuous at $x = 10$.

Summary Table

3	$\lim_{t\to\infty} S(t) = 500$; Maximum sales capacity
4(i)	10
4(ii)	Approaches 0
5(i)	∞ (unbounded)
5(ii)	82.44
6	Yes, continuous
7(i)	128.40
7(ii)	164.87
8	Yes, continuous