EXERCISE 13.1 - DERIVATIVES BY FIRST PRINCIPLE

Complete Solutions

Question 1: Find by definition, the derivatives w.r.t. 'x'

Definition of derivative:

$$f^{'}(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

(i) $\mathbf{f(x) = 2x^2+1}$

Step 1: $f(x+h) = 2(x+h)^2+1 = 2(x^2+2xh+h^2)+1 = 2x^2+4xh+2h^2+1$

Step 2: $f(x+h)-f(x) = (2x^2+4xh+2h^2+1)-(2x^2+1) = 4xh+2h^2$

Step 3: $\frac{f(x+h)-f(x)}{h} = \frac{4xh+2h^2}{h} = 4x+2h$

Step 4: $f^{'}(x) = \lim_{h\to 0}(4x+2h) = 4x$

Answer: $4x$

(ii) $\mathbf{f(x) = 2-\sqrt{x}}$

Step 1: $f(x+h)-f(x) = (2-\sqrt{x+h})-(2-\sqrt{x}) = -\sqrt{x+h}+\sqrt{x} = \sqrt{x}-\sqrt{x+h}$

Step 2: Rationalize:

$$\frac{\sqrt{x}-\sqrt{x+h}}{h}\times\frac{\sqrt{x}+\sqrt{x+h}}{\sqrt{x}+\sqrt{x+h}} = \frac{x-(x+h)}{h(\sqrt{x}+\sqrt{x+h})} = \frac{-h}{h(\sqrt{x}+\sqrt{x+h})} = \frac{-1}{\sqrt{x}+\sqrt{x+h}}$$

Step 3: $f^{'}(x) = \lim_{h\to 0}\frac{-1}{\sqrt{x}+\sqrt{x+h}} = \frac{-1}{2\sqrt{x}}$

Answer: $-\frac{1}{2\sqrt{x}}$

(iii) $\mathbf{f(x) = \frac{1}{\sqrt{x}}}$

Step 1: $f(x+h)-f(x) = \frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}} = \frac{\sqrt{x}-\sqrt{x+h}}{\sqrt{x}\sqrt{x+h}}$

Step 2: $\frac{f(x+h)-f(x)}{h} = \frac{\sqrt{x}-\sqrt{x+h}}{h\sqrt{x}\sqrt{x+h}}$

Step 3: Rationalize:

$$= \frac{(\sqrt{x}-\sqrt{x+h})(\sqrt{x}+\sqrt{x+h})}{h\sqrt{x}\sqrt{x+h}(\sqrt{x}+\sqrt{x+h})} = \frac{x-(x+h)}{h\sqrt{x}\sqrt{x+h}(\sqrt{x}+\sqrt{x+h})} = \frac{-h}{h\sqrt{x}\sqrt{x+h}(\sqrt{x}+\sqrt{x+h})} = \frac{-1}{\sqrt{x}\sqrt{x+h}(\sqrt{x}+\sqrt{x+h})}$$

Step 4: $f^{'}(x) = \lim_{h\to 0}\frac{-1}{\sqrt{x}\sqrt{x+h}(\sqrt{x}+\sqrt{x+h})} = \frac{-1}{x(2\sqrt{x})} = -\frac{1}{2x^{3/2}}$

Answer: $-\frac{1}{2x^{3/2}}$

(iv) $\mathbf{f(x) = x(x-3) = x^2-3x}$

Step 1: $f(x+h) = (x+h)^2-3(x+h) = x^2+2xh+h^2-3x-3h$

Step 2: $f(x+h)-f(x) = (x^2+2xh+h^2-3x-3h)-(x^2-3x) = 2xh+h^2-3h$

Step 3: $\frac{f(x+h)-f(x)}{h} = 2x + h - 3$

Step 4: $f^{'}(x) = \lim_{h\to 0}(2x + h - 3) = 2x - 3$

Answer: $2x - 3$

Question 2: Find $\frac{\mathbf{dy}}{\mathbf{dx}}$ from first principle and gradient at given point

(i) $\mathbf{y} = \sqrt{\mathbf{x+2}}$ at $\mathbf{x = 6}$

Step 1: $f(x+h) - f(x) = \sqrt{x+h+2} - \sqrt{x+2}$

Step 2: Rationalize:

$$\frac{\sqrt{x+h+2}-\sqrt{x+2}}{h} \times \frac{\sqrt{x+h+2}+\sqrt{x+2}}{\sqrt{x+h+2}+\sqrt{x+2}} = \frac{(x+h+2)-(x+2)}{h(\sqrt{x+h+2}+\sqrt{x+2})} = \frac{h}{h(\sqrt{x+h+2}+\sqrt{x+2})} = \frac{1}{\sqrt{x+h+2}+\sqrt{x+}}$$

Step 3: $f^{'}(x) = \lim_{h\to 0}\frac{1}{\sqrt{x+h+2}+\sqrt{x+2}} = \frac{1}{2\sqrt{x+2}}$

Step 4: At $x = 6$: $f^{'}(6) = \frac{1}{2\sqrt{8}} = \frac{1}{4\sqrt{2}}$

Answer: $\frac{dy}{dx} = \frac{1}{2\sqrt{x+2}}$, Gradient at $x = 6$ is $\frac{1}{4\sqrt{2}}$

(ii) $\mathbf{y} = \frac{\mathbf{1}}{\sqrt{\mathbf{x+a}}}$ at $\mathbf{x = a}$

Step 1: $f(x+h) - f(x) = \frac{1}{\sqrt{x+h+a}} - \frac{1}{\sqrt{x+a}} = \frac{\sqrt{x+a}-\sqrt{x+h+a}}{\sqrt{x+a}\sqrt{x+h+a}}$

Step 2: $\frac{f(x+h)-f(x)}{h} = \frac{\sqrt{x+a}-\sqrt{x+h+a}}{h\sqrt{x+a}\sqrt{x+h+a}}$

Step 3: Rationalize:

$$= \frac{(\sqrt{x+a}-\sqrt{x+h+a})(\sqrt{x+a}+\sqrt{x+h+a})}{h\sqrt{x+a}\sqrt{x+h+a}(\sqrt{x+a}+\sqrt{x+h+a})} = \frac{(x+a)-(x+h+a)}{h\sqrt{x+a}\sqrt{x+h+a}(\sqrt{x+a}+\sqrt{x+h+a})} = \frac{-h}{h\sqrt{x+a}\sqrt{x+h+a}(\sqrt{x+}}$$

Step 4: $f^{'}(x) = \frac{-1}{(x+a)(2\sqrt{x+a})} = -\frac{1}{2(x+a)^{3/2}}$

Step 5: At $x = a$: $f^{'}(a) = -\frac{1}{2(2a)^{3/2}}$

Answer: $\frac{dy}{dx} = -\frac{1}{2(x+a)^{3/2}}$, Gradient at $x = a$ is $-\frac{1}{2(2a)^{3/2}}$

Question 3

(i) Derivative of $\mathbf{x^3}$ at $\mathbf{x = 8}$ from first principle

Step 1: $f(x) = x^3$, $f(x+h) = (x+h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$

Step 2: $f(x+h) - f(x) = 3x^2h + 3xh^2 + h^3$

Step 3: $\frac{f(x+h)-f(x)}{h} = 3x^2 + 3xh + h^2$

Step 4: $f^{'}(x) = \lim_{h\to 0}(3x^2 + 3xh + h^2) = 3x^2$

Step 5: At $x = 8$: $f^{'}(8) = 3(8)^2 = 192$

Answer: 192

(ii) Derivative of $\mathbf{x^2+2x+3}$ by definition

Step 1: $f(x) = x^2 + 2x + 3$

Step 2: $f(x+h) = (x+h)^2 + 2(x+h) + 3 = x^2 + 2xh + h^2 + 2x + 2h + 3$

Step 3: $f(x+h) - f(x) = 2xh + h^2 + 2h$

Step 4: $\frac{f(x+h)-f(x)}{h} = 2x + h + 2$

Step 5: $f^{'}(x) = \lim_{h\to 0}(2x + h + 2) = 2x + 2$

Answer: $2x + 2$

Question 4: Find from first principle

(i) $\mathbf{f(x) = (3x - 2)^2}$

Step 1: $f(x+h) = (3x + 3h - 2)^2 = (3x-2)^2 + 6h(3x-2) + 9h^2$

Step 2: $f(x+h) - f(x) = 6h(3x-2) + 9h^2$

Step 3: $\frac{f(x+h)-f(x)}{h} = 6(3x-2) + 9h$

Step 4: $f^{'}(x) = \lim_{h\to 0}[6(3x-2) + 9h] = 6(3x-2)$

Answer: $6(3x-2)$

(ii) $\mathbf{f(x) = (2x + 3)^5}$

Step 1: Using binomial expansion:

$$(2x + 2h + 3)^5 - (2x+3)^5$$

Step 2: First term of expansion:

$$= 5(2x+3)^4(2h) + \text{higher terms}$$

Step 3: $\frac{f(x+h)-f(x)}{h} = 10(2x+3)^4 + \text{terms with } h$

Step 4: $f^{'}(x) = \lim_{h\to 0}[10(2x+3)^4 + \text{terms with } h] = 10(2x+3)^4$

Answer: $10(2x+3)^4$

(iii) $\mathbf{f(a) = (a + b)^7}$ w.r.t. a

Step 1: $f(a+h) = (a+h+b)^7$

Step 2: Using binomial expansion first term:

$$(a+b+h)^7 - (a+b)^7 = 7(a+b)^6 h + \text{higher terms}$$

Step 3: $f^{'}(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = 7(a+b)^6$

Answer: $7(a+b)^6$

Question 5: Gradient and tangent line at $\mathbf{x = 2}$

$$\mathbf{y = 3x^2 - 4x + 1}$$

Step 1: $f^{'}(x) = 6x - 4$

Step 2: At $x = 2$, gradient $= f'(2) = 6(2) - 4 = 8$

Step 3: Point at $x = 2$: $y = 3(4) - 4(2) + 1 = 12 - 8 + 1 = 5 \to (2, 5)$

Step 4: Tangent line: $y - 5 = 8(x - 2) \to y = 8x - 16 + 5 = 8x - 11$

Answer: Gradient $= 8$, Tangent: $y = 8x - 11$

Question 6: Tangent line at $x = -1$

$$\mathbf{f(x) = 2x^3 + x}$$

Step 1: $f'(x) = 6x^2 + 1$

Step 2: At $x = -1$, gradient $= f'(-1) = 6(-1)^2 + 1 = 7$

Step 3: Point: $f(-1) = 2(-1)^3 + (-1) = -2 - 1 = -3 \to (-1, -3)$

Step 4: Tangent: $y + 3 = 7(x + 1) \to y = 7x + 7 - 3 = 7x + 4$

Answer: Tangent: $y = 7x + 4$

Question 7: Point of tangency and tangent line at $x = 1$

$$\mathbf{f(x) = x^3 - 2x + 1}$$

Step 1: Point: $f(1) = 1 - 2 + 1 = 0 \to (1, 0)$

Step 2: $f'(x) = 3x^2 - 2$

Step 3: Gradient at $x = 1$: $f'(1) = 3(1)^2 - 2 = 1$

Step 4: Tangent: $y - 0 = 1(x - 1) \to y = x - 1$

Answer: Point $(1, 0)$, Tangent: $y = x - 1$

Question 8: Gradient of $\mathbf{f(x) = 3x^2 + 2x}$ at $x = 1$

Step 1: $f'(x) = 6x + 2$

Step 2: $f'(1) = 6(1) + 2 = 8$

Answer: 8

Question 9: Gradient and tangent line to $\mathbf{f(x) = \sqrt{x}}$ at $x = 9$

Step 1: $f'(x) = \frac{1}{2\sqrt{x}}$

Step 2: At $x = 9$: gradient $= \frac{1}{2\sqrt{9}} = \frac{1}{6}$

Step 3: Point: $f(9) = \sqrt{9} = 3 \to (9, 3)$

Step 4: Tangent: $y - 3 = \frac{1}{6}(x - 9) \to y = \frac{1}{6}x - \frac{3}{2} + 3 = \frac{1}{6}x + \frac{3}{2}$

Answer: Gradient $= \frac{1}{6}$, Tangent: $y = \frac{1}{6}x + \frac{3}{2}$

Question 10: Car position

$$\mathbf{s(t) = 2t^3 - 3t^2 + t}$$

(i) Average velocity over $[1, 4]$

Step 1: $s(4) = 2(64) - 3(16) + 4 = 128 - 48 + 4 = 84$
Step 2: $s(1) = 2(1) - 3(1) + 1 = 2 - 3 + 1 = 0$

Step 3: Average velocity $= \frac{s(4)-s(1)}{4-1} = \frac{84-0}{3} = 28$ km/h

Answer: 28 km/h

(ii) Instantaneous velocity at t = 2

Step 1: $s'(t) = 6t^2 - 6t + 1$

Step 2: $s'(2) = 6(4) - 6(2) + 1 = 24 - 12 + 1 = 13$ km/h

Answer: 13 km/h

Question 11: Stone thrown upwards

$$\mathbf{s(t) = -16t^2 + 32t + 10}$$

Step 1: $s'(t) = -32t + 32$

Step 2: $s'(1) = -32(1) + 32 = 0$ ft/s

Answer: 0 ft/s (at maximum height)

Question 12: Temperature rate of change

$$\mathbf{T(t) = -t^2 + 12t + 10}$$

Step 1: $T'(t) = -2t + 12$

Step 2: $T'(2) = -2(2) + 12 = 8$ °C/hour

Answer: 8°C/hour

Summary Table

1(i)	$4x$
1(ii)	$-\frac{1}{2\sqrt{x}}$
1(iii)	$-\frac{1}{2x^{3/2}}$
1(iv)	$2x - 3$
2(i)	$\frac{1}{2\sqrt{x+2}}, \frac{1}{4\sqrt{2}}$
2(ii)	$-\frac{1}{2(x+a)^{3/2}}, -\frac{1}{2(2a)^{3/2}}$
3(i)	192
3(ii)	$2x + 2$
4(i)	$6(3x - 2)$
4(ii)	$10(2x + 3)^4$
4(iii)	$7(a + b)^6$
5	Gradient=8, $y = 8x - 11$
6	$y = 7x + 4$
7	$(1, 0), y = x - 1$
8	8
9	$\frac{1}{6}, y = \frac{1}{6}x + \frac{3}{2}$
10(i)	28 km/h
10(ii)	13 km/h
11	0 ft/s

12	8°C/hour