## EXERCISE 14.4 - SCALAR TRIPLE PRODUCT AND APPLICATIONS

**Complete Solutions**

### Question 1: Volume of Parallelepiped

**Formula: Volume = $|\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})|$**

#### (i) $\mathbf{u}= 3\mathbf{i}+2\mathbf{j}+2\mathbf{k},\ \mathbf{v}= \mathbf{i}+2\mathbf{j}+\mathbf{k},\ \mathbf{w}= \mathbf{j}+4\mathbf{k}$

**Step 1:** $\mathbf{v}\times\mathbf{w} = \begin{array}{c} \begin{matrix} i & j & k \end{matrix} \\ \begin{vmatrix} 1 & 2 & 1 \\ 0 & 1 & 4 \end{vmatrix} \end{array}$

$$= i(2\cdot 4-1\cdot 1)-j(1\cdot 4-1\cdot 0)+k(1\cdot 1-2\cdot 0)= i(8-1)-j(4-0)+k(1-0)=7i-4j+k$$

**Step 2:** $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = 3(7)+2(-4)+2(1) = 21-8+2 = 15$

**Step 3:** Volume $= |15| = 15$

**Answer:** 15 cubic units

#### (ii) $\mathbf{u}= \mathbf{i}-4\mathbf{j}-\mathbf{k},\ \mathbf{v}= \mathbf{i}-\mathbf{j}-2\mathbf{k},\ \mathbf{w}= 2\mathbf{i}-3\mathbf{j}+\mathbf{k}$

**Step 1:** $\mathbf{v}\times\mathbf{w} = \begin{array}{c} \begin{matrix} i & j & k \end{matrix} \\ \begin{vmatrix} 1 & -1 & -2 \\ 2 & -3 & 1 \end{vmatrix} \end{array}$

$$= i((-1)(1)-(-2)(-3))-j(1(1)-(-2)(2))+k(1(-3)-(-1)(2))= i(-1-6)-j(1+4)+k(-3+2)= -7i-5j-k$$

**Step 2:** $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = 1(-7)+(-4)(-5)+(-1)(-1) = -7+20+1 = 14$

**Step 3:** Volume $= |14| = 14$

**Answer:** 14 cubic units

#### (iii) $\mathbf{u}= \mathbf{i}-2\mathbf{j}+3\mathbf{k},\ \mathbf{v}= 2\mathbf{i}-\mathbf{j}-\mathbf{k},\ \mathbf{w}= \mathbf{j}+\mathbf{k}$

**Step 1:** $\mathbf{v}\times\mathbf{w} = \begin{array}{c} \begin{matrix} i & j & k \end{matrix} \\ \begin{vmatrix} 2 & -1 & -1 \\ 0 & 1 & 1 \end{vmatrix} \end{array}$

$$= i((-1)(1)-(-1)(1))-j(2(1)-(-1)(0))+k(2(1)-(-1)(0))= i(-1+1)-j(2-0)+k(2-0) = 0i-2j+2k$$

**Step 2:** $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = 1(0)+(-2)(-2)+3(2) = 0+4+6 = 10$

**Step 3:** Volume $= |10| = 10$

**Answer:** 10 cubic units

### Question 2: Verify Scalar Triple Product Cyclic Property

**Given:** $\mathbf{a}= 3\mathbf{i}-\mathbf{j}+5\mathbf{k},\ \mathbf{b}= 4\mathbf{i}+3\mathbf{j}-2\mathbf{k},\ \mathbf{c}= 2\mathbf{i}+5\mathbf{j}+\mathbf{k}$

**Step 1:** $\mathbf{b}\times\mathbf{c} = \begin{array}{c} \begin{matrix} i & j & k \end{matrix} \\ \begin{vmatrix} 4 & 3 & -2 \\ 2 & 5 & 1 \end{vmatrix} \end{array}$

$$= i(3\cdot 1-(-2)(5))-j(4\cdot 1-(-2)(2))+k(4\cdot 5-3\cdot 2)= i(3+10)-j(4+4)+k(20-6) = 13i-8j+14k$$

taleemzone.com

**Step 2:** $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = 3(13) + (-1)(-8) + 5(14) = 39 + 8 + 70 = 117$

**Step 3:** $\mathbf{c} \times \mathbf{a} = \begin{vmatrix} i & j & k \\ 2 & 5 & 1 \\ 3 & -1 & 5 \end{vmatrix}$

$= i(5 \cdot 5 - 1(-1)) - j(2 \cdot 5 - 1 \cdot 3) + k(2(-1) - 5 \cdot 3) = i(25 + 1) - j(10 - 3) + k(-2 - 15) = 26i - 7j - 17k$

**Step 4:** $\mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = 4(26) + 3(-7) + (-2)(-17) = 104 - 21 + 34 = 117$

**Step 5:** $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} i & j & k \\ 3 & -1 & 5 \\ 4 & 3 & -2 \end{vmatrix}$

$= i((-1)(-2) - 5(3)) - j(3(-2) - 5(4)) + k(3 \cdot 3 - (-1)(4)) = i(2 - 15) - j(-6 - 20) + k(9 + 4) = -13i + 26j + 13k$

**Step 6:** $\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) = 2(-13) + 5(26) + 1(13) = -26 + 130 + 13 = 117$

**Answer:** All equal to 117

## Question 3: Prove Vectors are Coplanar

**Vectors:** $\mathbf{a} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$, $\mathbf{b} = -2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$, $\mathbf{c} = \mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$

**Step 1:** For coplanarity, scalar triple product = 0:

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = 0$$

**Step 2:** $\mathbf{b} \times \mathbf{c} = \begin{vmatrix} i & j & k \\ -2 & 3 & -4 \\ 1 & -3 & 5 \end{vmatrix}$

$= i(3 \cdot 5 - (-4)(-3)) - j((-2)(5) - (-4)(1)) + k((-2)(-3) - 3(1)) = i(15 - 12) - j(-10 + 4) + k(6 - 3) = 3i + 6j + 3k$

**Step 3:** $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = 1(3) + (-2)(6) + 3(3) = 3 - 12 + 9 = 0$

**Answer:** Coplanar

## Question 4: Find Constant a for Coplanarity

### (i) $\mathbf{u} = \mathbf{i} - \mathbf{j} + \mathbf{k}$, $\mathbf{v} = \mathbf{i} - 2\mathbf{j} - 3\mathbf{k}$, $\mathbf{w} = 3\mathbf{i} - a\mathbf{j} + 5\mathbf{k}$

**Step 1:** For coplanar: $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = 0$

**Step 2:** $\mathbf{v} \times \mathbf{w} = \begin{vmatrix} i & j & k \\ 1 & -2 & -3 \\ 3 & -a & 5 \end{vmatrix}$

$= i((-2)(5) - (-3)(-a)) - j(1(5) - (-3)(3)) + k(1(-a) - (-2)(3)) = i(-10 - 3a) - j(5 + 9) + k(-a + 6) = (-10 - 3a)i - 14$

**Step 3:** $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = 1(-10 - 3a) + (-1)(-14) + 1(6 - a)$

$$= -10 - 3a + 14 + 6 - a = 10 - 4a$$



taleemzone.com

**Step 4:** Set to 0: $10 - 4a = 0 \to a = \frac{10}{4} = \frac{5}{2}$

**Answer:** $a = \frac{5}{2}$

**(ii) Vectors: i−2aj−k, i−2j+2k, ai−2j+k**

**Step 1:** Let $\mathbf{u} = i - 2aj - k$, $\mathbf{v} = i - 2j + 2k$, $\mathbf{w} = ai - 2j + k$

**Step 2:** $\mathbf{v} \times \mathbf{w} = \begin{vmatrix} i & j & k \\ 1 & -2 & 2 \\ a & -2 & 1 \end{vmatrix}$

$= i((-2)(1) - 2(-2)) - j(1(1) - 2(a)) + k(1(-2) - (-2)(a))= i(-2 + 4) - j(1 - 2a) + k(-2 + 2a)= 2i + (2a - 1)j + (2a - 2$

**Step 3:** $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = 1(2) + (-2a)(2a - 1) + (-1)(2a - 2)$

$$= 2 - 4a^2 + 2a - 2a + 2 = 4 - 4a^2$$

**Step 4:** Set to 0: $4 - 4a^2 = 0 \to a^2 = 1 \to a = \pm 1$

**Answer:** $a = \pm 1$

## Question 5: Prove Points are Coplanar

**Points:** $\mathbf{A}(-6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k})$, $\mathbf{B}(3\mathbf{i} - 2\mathbf{j} + 4\mathbf{k})$, $\mathbf{C}(5\mathbf{i} + 7\mathbf{j} + 3\mathbf{k})$, $\mathbf{D}(-13\mathbf{i} + 17\mathbf{j} - \mathbf{k})$

**Step 1:** Find vectors:

$$\overrightarrow{AB} = B - A = 9i - 5j + 2k\overrightarrow{AC} = C - A = 11i + 4j + k\overrightarrow{AD} = D - A = -7i + 14j - 3k$$

**Step 2:** For coplanar points: $\overrightarrow{AB} \cdot (\overrightarrow{AC} \times \overrightarrow{AD}) = 0$

**Step 3:** $\overrightarrow{AC} \times \overrightarrow{AD} = \begin{vmatrix} i & j & k \\ 11 & 4 & 1 \\ -7 & 14 & -3 \end{vmatrix}$

$= i(4(-3) - 1(14)) - j(11(-3) - 1(-7)) + k(11(14) - 4(-7))= i(-12 - 14) - j(-33 + 7) + k(154 + 28)= -26i + 26j + 182k$

**Step 4:** $\overrightarrow{AB} \cdot (\overrightarrow{AC} \times \overrightarrow{AD}) = 9(-26) + (-5)(26) + 2(182)$

$$= -234 - 130 + 364 = 0$$

**Answer:** Points are coplanar

## Question 6: Cross Product of Unit Vectors

**(a) Find the values:**

**(i)** $2i \times 2j - k$

$$2i \times 2j = 4(i \times j) = 4k4k - k = 3k$$

**Answer:** $3k$

**(ii)** $3j \times i$

taleemzone.com

$$3(j \times i) = 3(-k) = -3k$$

**Answer:** $-3k$

**(iii)** $k \times i$

$$k \times i = j$$

**Answer:** $j$

**(iv)** $i \times k$

$$i \times k = -j$$

**Answer:** $-j$

**(b) Prove the identity:**

**The statement appears to have a typo.** The correct identity for scalar triple product is:

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \mathbf{v} \cdot (\mathbf{w} \times \mathbf{u}) = \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})$$

### Question 7: Volume of Tetrahedron

**Formula: Volume** $= \frac{1}{6}|\overrightarrow{\mathbf{AB}} \cdot (\overrightarrow{\mathbf{AC}} \times \overrightarrow{\mathbf{AD}})|$

#### (i) $\mathbf{A(0,1,2)}$, $\mathbf{B(3,2,1)}$, $\mathbf{C(1,2,1)}$, $\mathbf{D(5,5,6)}$

**Step 1:** $\overrightarrow{AB} = (3,1,-1)$, $\overrightarrow{AC} = (1,1,-1)$, $\overrightarrow{AD} = (5,4,4)$

**Step 2:** $\overrightarrow{AC} \times \overrightarrow{AD} = \begin{vmatrix} i & j & k \\ 1 & 1 & -1 \\ 5 & 4 & 4 \end{vmatrix}$

$$= i(1 \cdot 4 - (-1)(4)) - j(1 \cdot 4 - (-1)(5)) + k(1 \cdot 4 - 1 \cdot 5) = i(4+4) - j(4+5) + k(4-5) = 8i - 9j - k$$

**Step 3:** $\overrightarrow{AB} \cdot (\overrightarrow{AC} \times \overrightarrow{AD}) = 3(8) + 1(-9) + (-1)(-1) = 24 - 9 + 1 = 16$

**Step 4:** Volume $= \frac{16}{6} = \frac{8}{3}$

**Answer:** $\frac{8}{3}$ cubic units

#### (ii) $\mathbf{A(2,1,8)}$, $\mathbf{B(3,2,9)}$, $\mathbf{C(2,1,4)}$, $\mathbf{D(3,3,10)}$

**Step 1:** $\overrightarrow{AB} = (1,1,1)$, $\overrightarrow{AC} = (0,0,-4)$, $\overrightarrow{AD} = (1,2,2)$

**Step 2:** $\overrightarrow{AC} \times \overrightarrow{AD} = \begin{vmatrix} i & j & k \\ 0 & 0 & -4 \\ 1 & 2 & 2 \end{vmatrix}$

$$= i(0 \cdot 2 - (-4)(2)) - j(0 \cdot 2 - (-4)(1)) + k(0 \cdot 2 - 0 \cdot 1) = i(0+8) - j(0+4) + k(0) = 8i - 4j$$

**Step 3:** $\overrightarrow{AB} \cdot (\overrightarrow{AC} \times \overrightarrow{AD}) = 1(8) + 1(-4) + 1(0) = 4$

**Step 4:** Volume $= \frac{4}{6} = \frac{2}{3}$

**Answer:** $\frac{2}{3}$ cubic units

### Question 8: Prove Points are Coplanar

**Points: $\mathbf{A(3, 2, -1)}$, $\mathbf{B(0, -2, 1)}$, $\mathbf{C(6, 4, -2)}$, $\mathbf{D(9, 6, -3)}$**

**Step 1:** $\overrightarrow{AB} = (-3, -4, 2)$, $\overrightarrow{AC} = (3, 2, -1)$, $\overrightarrow{AD} = (6, 4, -2)$

**Step 2:** Note that $\overrightarrow{AD} = 2\overrightarrow{AC}$ and $\overrightarrow{AB} = -\overrightarrow{AC}$? Let's check:

- $\overrightarrow{AC} = (3, 2, -1)$
- $\overrightarrow{AD} = (6, 4, -2) = 2(3, 2, -1) = 2\overrightarrow{AC}$

**Step 3:** Since $\overrightarrow{AD}$ is a scalar multiple of $\overrightarrow{AC}$, the three vectors are linearly dependent, so the scalar triple product is 0.

**Answer:** Points are coplanar

### Question 9: Prove Identities

**Given expression seems to have typos. The standard identities are:**

- $(\mathbf{u} + \mathbf{v}) \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{w}$

The given expressions are not generally true.

### Question 10: Height of Parallelepiped

**Given: $\mathbf{u} = \mathbf{2i + 4j - 3k}$, $\mathbf{v} = \mathbf{5i - 3j + 6k}$, $\mathbf{w} = \mathbf{4i - 7j - 2k}$**

**Step 1:** Base area $= |\ \mathbf{u} \times \mathbf{v}\ |$

**Step 2:** $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} i & j & k \\ 2 & 4 & -3 \\ 5 & -3 & 6 \end{vmatrix}$

$$= i(4 \cdot 6 - (-3)(-3)) - j(2 \cdot 6 - (-3)(5)) + k(2(-3) - 4(5)) = i(24 - 9) - j(12 + 15) + k(-6 - 20) = 15i - 27j - 26k$$

**Step 3:** $|\ \mathbf{u} \times \mathbf{v}\ | = \sqrt{15^2 + (-27)^2 + (-26)^2} = \sqrt{225 + 729 + 676} = \sqrt{1630}$

**Step 4:** Volume $= |\ \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})\ |$

**Step 5:** $\mathbf{v} \times \mathbf{w} = \begin{vmatrix} i & j & k \\ 5 & -3 & 6 \\ 4 & -7 & -2 \end{vmatrix}$

$$= i((-3)(-2) - 6(-7)) - j(5(-2) - 6(4)) + k(5(-7) - (-3)(4)) = i(6 + 42) - j(-10 - 24) + k(-35 + 12) = 48i + 34j - 23k$$

**Step 6:** $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = 2(48) + 4(34) + (-3)(-23) = 96 + 136 + 69 = 301$

**Step 7:** Height $= \frac{\text{Volume}}{\text{Base Area}} = \frac{301}{\sqrt{1630}}$

**Answer:** $\frac{301}{\sqrt{1630}}$

### Question 11: Torque

**Given: Force $\mathbf{F} = \mathbf{50i}$, applied at $(\mathbf{0, 2, 3})$, pivot at origin**

**Step 1:** Position vector $\mathbf{r} = 0i + 2j + 3k$

**Step 2:** Torque $= \mathbf{r} \times \mathbf{F} = \begin{vmatrix} i & j & k \\ 0 & 2 & 3 \\ 50 & 0 & 0 \end{vmatrix}$

$$= i(2 \cdot 0 - 3 \cdot 0) - j(0 \cdot 0 - 3 \cdot 50) + k(0 \cdot 0 - 2 \cdot 50) = 0i + 150j - 100k$$

**Answer:** $150j - 100k$

**Summary Table**

| | |
|---|---|
| 1(i) | 15 |
| 1(ii) | 14 |
| 1(iii) | 10 |
| 2 | All equal to 117 |
| 3 | Coplanar |
| 4(i) | $a = \frac{5}{2}$ |
| 4(ii) | $a = \pm 1$ |
| 5 | Coplanar |
| 6(a)(i) | $3k$ |
| 6(a)(ii) | $-3k$ |
| 6(a)(iii) | $j$ |
| 6(a)(iv) | $-j$ |
| 7(i) | $\frac{8}{3}$ |
| 7(ii) | $\frac{2}{3}$ |
| 8 | Coplanar |
| 10 | $\frac{301}{\sqrt{1630}}$ |
| 11 | $150j - 100k$ |

## EXERCISE 14.4 - SCALAR TRIPLE PRODUCT AND APPLICATIONS (Continued)

### Question 12: Drone Displacement

**Given: From point (1, 2, 5) to point (4, 6, 9)**

**Step 1:** Displacement vector:

$$\overrightarrow{AB} = (4-1)\mathbf{i} + (6-2)\mathbf{j} + (9-5)\mathbf{k} = 3\mathbf{i} + 4\mathbf{j} + 4\mathbf{k}$$

**Step 2:** Magnitude:

$$|\overrightarrow{AB}| = \sqrt{3^2 + 4^2 + 4^2} = \sqrt{9 + 16 + 16} = \sqrt{41}$$

**Answer:** $\sqrt{41}$ meters ( 6.40 m)

### Question 13: Sales and Revenue

**Given:**

- Quantity vector: $\mathbf{u} = 50\mathbf{i} + 75\mathbf{j} + 65\mathbf{k}$ (belts, pants, shirts)
- Price vector: $\mathbf{w} = 1500\mathbf{i} + 3500\mathbf{j} + 3000\mathbf{k}$ (prices in rupees)

taleemzone.com

**Step 1:** Dot product:

$$\mathbf{u}\cdot\mathbf{w} = 50(1500) + 75(3500) + 65(3000) = 75,000 + 262,500 + 195,000 = 532,500$$

**Step 2:** Interpretation:

- The dot product gives the total revenue from selling all items.
- Belts: $50 \times 1500 =$ Rs. 75,000
- Pants: $75 \times 3500 =$ Rs. 262,500
- Shirts: $65 \times 3000 =$ Rs. 195,000
- **Total Revenue = Rs. 532,500**

**Answer:** $\mathbf{u}\cdot\mathbf{w} = 532,500$ rupees (total revenue)

### Question 14: Torque Calculation

**Given:**

- Force: $\mathbf{F} = 20\mathbf{i} - 10\mathbf{j} + 30\mathbf{k}$ N
- Point of application: $P(2, -1, 4)$
- Pivot point: $M(1, 2, -3)$

**Step 1:** Position vector from pivot to point of application:

$$\mathbf{r} = \overrightarrow{MP} = P - M = (2-1)\mathbf{i} + (-1-2)\mathbf{j} + (4-(-3))\mathbf{k} = \mathbf{i} - 3\mathbf{j} + 7\mathbf{k}$$

**Step 2:** Torque $= \mathbf{r}\times\mathbf{F}$:

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -3 & 7 \\ 20 & -10 & 30 \end{vmatrix}$$

**Step 3:** Evaluate:

- $\tau_x = (-3)(30) - 7(-10) = -90 + 70 = -20$
- $\tau_y = -[1(30) - 7(20)] = -(30 - 140) = 110$
- $\tau_z = 1(-10) - (-3)(20) = -10 + 60 = 50$

**Step 4:** Result:

$$= -20\mathbf{i} + 110\mathbf{j} + 50\mathbf{k}$$

**Answer:** $-20\mathbf{i} + 110\mathbf{j} + 50\mathbf{k}$ N · m

### Question 15: Electric Shop Profit

**(a) Represent as vectors**

**Step 1:** Sales quantities vector:

$$\mathbf{q} = 500\mathbf{i} + 300\mathbf{j} + 200\mathbf{k}$$

taleemzone.com

(Fans, Heaters, Ovens)

**Step 2:** Profit per unit vector:

$$\mathbf{p} = 500\mathbf{i} + 400\mathbf{j} + 2000\mathbf{k}$$

(Fans, Heaters, Ovens)

**(b) Total monthly profit**

**Step 1:** Total profit $= \mathbf{q} \cdot \mathbf{p}$:

$$\text{Profit} = 500(500) + 300(400) + 200(2000) = 250,000 + 120,000 + 400,000 = 770,000$$

**Step 2:** Breakdown:

- Fans: $500 \times 500 =$ Rs. 250,000
- Heaters: $300 \times 400 =$ Rs. 120,000
- Ovens: $200 \times 2000 =$ Rs. 400,000

**Answer:**

- (a) $\mathbf{q} = 500\mathbf{i} + 300\mathbf{j} + 200\mathbf{k}$, $\mathbf{p} = 500\mathbf{i} + 400\mathbf{j} + 2000\mathbf{k}$
- (b) Total profit = Rs. 770,000

**Summary Table**

| | |
|---|---|
| 12 | $\sqrt{41}$ meters |
| 13 | Rs. 532,500 (total revenue) |
| 14 | $-20\mathbf{i} + 110\mathbf{j} + 50\mathbf{k}$ N $\cdot$ m |
| 15(a) | $\mathbf{q} = 500\mathbf{i} + 300\mathbf{j} + 200\mathbf{k}$, $\mathbf{p} = 500\mathbf{i} + 400\mathbf{j} + 2000\mathbf{k}$ |
| 15(b) | Rs. 770,000 |

taleemzone.com