EXERCISE 2.1 (Continued) - COMPLETE SOLUTIONS

Q.1

Given that:

(a) $\mathbf{f(x) = x^2 - 1}$

(b) $\mathbf{f(x) = \sqrt{2x + 3}}$

Find:

(i) $\mathbf{f(-3)}$

For (a) $\mathbf{f(x) = x^2 - 1}$**:**

$$f(-3) = (-3)^2 - 1 = 9 - 1 = 8$$

Answer: $f(-3) = 8$

For (b) $\mathbf{f(x) = \sqrt{2x + 3}}$**:**

$$f(-3) = \sqrt{2(-3) + 3} = \sqrt{-6 + 3} = \sqrt{-3}$$

Since the square root of a negative number is not defined in real numbers:

Answer: $f(-3)$ is undefined in $\mathbb{R}$

(ii) $\mathbf{f(0)}$

For (a) $\mathbf{f(x) = x^2 - 1}$**:**

$$f(0) = (0)^2 - 1 = 0 - 1 = -1$$

Answer: $f(0) = -1$

For (b) $\mathbf{f(x) = \sqrt{2x + 3}}$**:**

$$f(0) = \sqrt{2(0) + 3} = \sqrt{0 + 3} = \sqrt{3}$$

Answer: $f(0) = \sqrt{3}$

(iii) $\mathbf{f(x - 2)}$

For (a) $\mathbf{f(x) = x^2 - 1}$**:**

Replace x with $x - 2$:

$$f(x-2) = (x-2)^2 - 1 = x^2 - 4x + 4 - 1 = x^2 - 4x + 3$$

Answer: $f(x - 2) = x^2 - 4x + 3$

For (b) $\mathbf{f(x) = \sqrt{2x + 3}}$**:**

Replace x with $x - 2$:

$$f(x-2) = \sqrt{2(x-2) + 3} = \sqrt{2x - 4 + 3} = \sqrt{2x - 1}$$

Answer: $f(x-2)=\sqrt{2x-1}$

(iv) $\mathbf{f(x^2+3)}$

For (a) $\mathbf{f(x)=x^2-1}$**:**

Replace x with x^2+3:

$$f(x^2+3)=(x^2+3)^2-1=x^4+6x^2+9-1=x^4+6x^2+8$$

Answer: $f(x^2+3)=x^4+6x^2+8$

For (b) $\mathbf{f(x)=\sqrt{2x+3}}$**:**

Replace x with x^2+3:

$$f(x^2+3)=\sqrt{2(x^2+3)+3}=\sqrt{2x^2+6+3}=\sqrt{2x^2+9}$$

Answer: $f(x^2+3)=\sqrt{2x^2+9}$

Q.2

Find $\frac{f(a+h)-f(a)}{h}$ and simplify where:

(i) $\mathbf{f(x)=4x+7}$

Solution:

First find $f(a+h)$:

$$f(a+h)=4(a+h)+7=4a+4h+7$$

Now find $f(a)$:

$$f(a)=4a+7$$

Compute the difference quotient:

$$\frac{f(a+h)-f(a)}{h}=\frac{(4a+4h+7)-(4a+7)}{h}=\frac{4a+4h+7-4a-7}{h}=\frac{4h}{h}=4$$

Answer: $\frac{f(a+h)-f(a)}{h}=4$

(ii) $\mathbf{f(x)=\sin x}$

Solution:

First find $f(a+h)$:

$$f(a+h)=\sin(a+h)$$

Now find $f(a)$:

$$f(a)=\sin a$$

Compute the difference quotient:

$$\frac{f(a+h)-f(a)}{h} = \frac{\sin(a+h)-\sin a}{h}$$

Using the trigonometric identity:

$$\sin(a+h)-\sin a = 2\cos\left(a+\frac{h}{2}\right)\sin\left(\frac{h}{2}\right)$$

So:

$$\frac{f(a+h)-f(a)}{h} = \frac{2\cos\left(a+\frac{h}{2}\right)\sin\left(\frac{h}{2}\right)}{h} = \cos\left(a+\frac{h}{2}\right)\cdot\frac{2\sin\left(\frac{h}{2}\right)}{h} = \cos\left(a+\frac{h}{2}\right)\cdot\frac{\sin\left(\frac{h}{2}\right)}{h/2}$$

Answer: $\frac{f(a+h)-f(a)}{h} = \cos\left(a+\frac{h}{2}\right)\cdot\frac{\sin(h/2)}{h/2}$

(iii) $\mathbf{f(x) = x^3 + x^2 - 1}$

Solution:

First find $f(a+h)$:

$$f(a+h) = (a+h)^3 + (a+h)^2 - 1 = (a^3 + 3a^2h + 3ah^2 + h^3) + (a^2 + 2ah + h^2) - 1 = a^3 + 3a^2h + 3ah^2 + h^3 + a^2 + 2ah + h^2$$

Now find $f(a)$:

$$f(a) = a^3 + a^2 - 1$$

Compute the difference quotient:

$$\frac{f(a+h)-f(a)}{h} = \frac{(a^3 + 3a^2h + 3ah^2 + h^3 + a^2 + 2ah + h^2 - 1) - (a^3 + a^2 - 1)}{h}$$

Cancel a^3, a^2, and -1:

$$= \frac{3a^2h + 3ah^2 + h^3 + 2ah + h^2}{h}$$

Factor out h:

$$= \frac{h(3a^2 + 3ah + h^2 + 2a + h)}{h} = 3a^2 + 3ah + h^2 + 2a + h$$

Answer: $\frac{f(a+h)-f(a)}{h} = 3a^2 + 3ah + h^2 + 2a + h$

(iv) $\mathbf{f(x) = \tan x}$

Solution:

First find $f(a+h)$:

$$f(a+h) = \tan(a+h)$$

Now find $f(a)$:

$$f(a) = \tan a$$

Compute the difference quotient:

$$\frac{f(a+h) - f(a)}{h} = \frac{\tan(a+h) - \tan a}{h}$$

Using the tangent subtraction identity:

$$\tan(a+h) - \tan a = \frac{\sin(a+h)}{\cos(a+h)} - \frac{\sin a}{\cos a} = \frac{\sin(a+h)\cos a - \sin a \cos(a+h)}{\cos(a+h)\cos a} = \frac{\sin[(a+h) - a]}{\cos(a+h)\cos a} = \frac{\sin h}{\cos(a+h)\cos a}$$

So:

$$\frac{f(a+h) - f(a)}{h} = \frac{\sin h}{h \cdot \cos(a+h)\cos a} = \frac{1}{\cos(a+h)\cos a} \cdot \frac{\sin h}{h}$$

Answer: $\frac{f(a+h)-f(a)}{h} = \frac{\sin h}{h \cdot \cos(a+h)\cos a}$

Or equivalently:

$$\boxed{\frac{f(a+h) - f(a)}{h} = \frac{1}{\cos(a+h)\cos a} \cdot \frac{\sin h}{h}}$$

EXERCISE 2.1 - COMPLETE SOLUTIONS

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Q.3

Express the following:

(a) The area A of a square as a function of its perimeter P.

Solution:

For a square with side length s:

- Perimeter: $P = 4s \quad s = \frac{P}{4}$
- Area: $A = s^2$

Substituting $s = \frac{P}{4}$:

$$A = \left(\frac{P}{4}\right)^2 = \frac{P^2}{16}$$

Answer: $A(P) = \frac{P^2}{16}$

(b) The circumference C of a circle as a function of its area A.

Solution:

For a circle with radius r:

- Area: $A = \pi r^2 \quad r = \sqrt{\frac{A}{\pi}}$
- Circumference: $C = 2\pi r$

Substituting $r = \sqrt{\frac{A}{\pi}}$:

$$C = 2\pi\sqrt{\frac{A}{\pi}} = 2\sqrt{\pi A}$$

Answer: $C(A) = 2\sqrt{\pi A}$

(c) The surface area S of a cube as a function of its volume V.

Solution:

For a cube with side length s:

- Volume: $V = s^3 \quad s = \sqrt[3]{V}$
- Surface area: $S = 6s^2$

Substituting $s = \sqrt[3]{V}$:

$$S = 6(\sqrt[3]{V})^2 = 6V^{2/3}$$

Answer: $S(V) = 6V^{2/3}$

Q.4

Find the domain and the range of the function g defined below:

(i) $\mathbf{g(x) = 5 - x}$

Solution:

This is a linear function defined for all real numbers.

Domain: $\mathbb{R}$ or $(-\infty, \infty)$

Range: $\mathbb{R}$ or $(-\infty, \infty)$

(ii) $\mathbf{g(x) = \sqrt{x + 2}}$

Solution:

For the square root to be defined, the radicand must be non-negative:

$$x + 2 \geq 0 \Rightarrow x \geq -2$$

Domain: $[-2, \infty)$

The square root function produces non-negative values.

Range: $[0, \infty)$

(iii) (g(x) = \begin{cases}

6x + 7, & x \leq -2 \
4 - 3x, & x > -2
\end{cases})

Solution:

This is a piecewise function.

Domain: $\mathbb{R}$ or $(-\infty, \infty)$

Range:

For $x \leq -2$: $g(x) = 6x + 7$

As $x \to -\infty$, $g(x) \to -\infty$. At $x = -2$: $g(-2) = 6(-2) + 7 = -12 + 7 = -5$

So for this piece: $(-\infty, -5]$

For $x > -2$: $g(x) = 4 - 3x$

At $x \to -2^+$, $g(x) \to 4 - 3(-2) = 4 + 6 = 10$. As $x \to \infty$, $g(x) \to -\infty$

So for this piece: $(-\infty, 10)$

Combining both pieces:

Range: $(-\infty, \infty)$ or $\mathbb{R}$

(iv) $\mathbf{g(x) = |x - 5|}$

Solution:

The absolute value function is defined for all real numbers.

Domain: $\mathbb{R}$ or $(-\infty, \infty)$

The absolute value produces non-negative values only.

Range: $[0, \infty)$

(v) $\mathbf{g(x) = \frac{x+2}{3-x}}$

Solution:

The function is undefined when the denominator is zero:

$$3 - x = 0 \Rightarrow x = 3$$

Domain: $\mathbb{R} \setminus \{3\}$ or $(-\infty, 3) \cup (3, \infty)$

Range:

Let $y = \frac{x+2}{3-x}$

Solve for x:

$$y(3 - x) = x + 23y - xy = x + 23y - 2 = x + xy3y - 2 = x(1 + y)x = \frac{3y - 2}{1 + y}$$

The range excludes the value that makes the denominator zero:

$$1 + y = 0 \Rightarrow y = -1$$

Range: $\mathbb{R} \setminus \{-1\}$ or $(-\infty, -1) \cup (-1, \infty)$

Q.5

Given $f(x) = x^3 - ax^2 + bx + 1$, if $f(2) = -3$ and $f(-1) = 0$, find the values of a and b.

Solution:

Given:

$$f(2) = -3$$

Substitute $x = 2$:

$$(2)^3 - a(2)^2 + b(2) + 1 = -38 - 4a + 2b + 1 = -39 - 4a + 2b = -3 - 4a + 2b = -12 - 2a + b = -6 \text{(i)}$$

Also:

$$f(-1) = 0$$

Substitute $x = -1$:

$$(-1)^3 - a(-1)^2 + b(-1) + 1 = 0 - 1 - a - b + 1 = 0 - a - b = 0 a + b = 0 \text{(ii)}$$

From (ii): $b = -a$

Substitute in (i):

$$-2a + (-a) = -6 - 3a = -6 a = 2$$

Then:

$$b = -2$$

Answer: $a = 2$, $b = -2$

Q.6

A stone falls from a height of 60m on the ground, the height h after x seconds is approximately given by $h(x) = 40 - 10x^2$.

(i) What is the height of stone when:

(a) $\mathbf{x = 1}$ sec?

$$h(1) = 40 - 10(1)^2 = 40 - 10 = 30 \text{ m}$$

Answer: 30 m

(b) $\mathbf{x = 1.5}$ sec?

$$h(1.5) = 40 - 10(1.5)^2 = 40 - 10(2.25) = 40 - 22.5 = 17.5 \text{ m}$$

Answer: 17.5 m

(c) $\mathbf{x = 1.7}$ sec?

$$h(1.7) = 40 - 10(1.7)^2 = 40 - 10(2.89) = 40 - 28.9 = 11.1 \text{ m}$$

Answer: 11.1 m

(ii) When does the stone strike the ground?

The stone strikes the ground when $h(x) = 0$:

$$40 - 10x^2 = 010x^2 = 40x^2 = 4x = 2 \text{ seconds(since time cannot be negative)}$$

Answer: After 2 seconds

Q.7

Consider the function $f(x) = 3x - 5$.

(i) Determine the domain and range of f(x).

Solution:

This is a linear function.

Domain: $\mathbb{R}$ or $(-\infty, \infty)$

Range: $\mathbb{R}$ or $(-\infty, \infty)$

(ii) Is the function f one-to-one? Justify your answer.

Solution:

To check if f is one-to-one (injective):

Assume $f(x_1) = f(x_2)$:

$$3x_1 - 5 = 3x_2 - 53x_1 = 3x_2x_1 = x_2$$

Since $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$, the function is one-to-one.

Answer: Yes, f is one-to-one.

(iii) Is the function f onto if the co-domain is all real numbers? Explain.

Solution:

A function is onto (surjective) if for every y in the co-domain, there exists x in the domain such that $f(x) = y$.

For $f(x) = 3x - 5$, given any $y \in \mathbb{R}$:

$$y = 3x - 53x = y + 5x = \frac{y+5}{3}$$

Since $x \in \mathbb{R}$ for every $y \in \mathbb{R}$, the function is onto.

Answer: Yes, f is onto.

Q.8

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \frac{2x-3}{x+1}$.

(i) Find the domain and range of f(x).

Solution:

The function is undefined when the denominator is zero:

$$x + 1 = 0 \Rightarrow x = -1$$

Domain: $\mathbb{R} \setminus \{-1\}$ or $(-\infty, -1) \cup (-1, \infty)$

Range:

Let $y = \frac{2x-3}{x+1}$

Solve for x:

$$y(x+1) = 2x - 3yx + y = 2x - 3yx - 2x = -3 - yx(y-2) = -3 - yx = \frac{-3-y}{y-2} = \frac{y+3}{2-y}$$

The range excludes the value that makes the denominator zero:

$$2 - y = 0 \Rightarrow y = 2$$

Range: $\mathbb{R} \setminus \{2\}$ or $(-\infty, 2) \cup (2, \infty)$

(ii) Determine whether f(x) is onto.

Solution:

For f to be onto, for every $y \in \mathbb{R}$, there must exist $x \in \mathbb{R} \setminus \{-1\}$ such that $f(x) = y$.

From part (i), we found that $y = 2$ is not in the range.

Since $2 \in \mathbb{R}$ but $2 \notin$ Range, the function is not onto.

Answer: No, f is not onto.

(iii) Prove that f(x) is one-to-one.

Solution:

Assume $f(x_1) = f(x_2)$:

$$\frac{2x_1 - 3}{x_1 + 1} = \frac{2x_2 - 3}{x_2 + 1}$$

Cross multiply:

$$(2x_1 - 3)(x_2 + 1) = (2x_2 - 3)(x_1 + 1)2x_1x_2 + 2x_1 - 3x_2 - 3 = 2x_1x_2 + 2x_2 - 3x_1 - 32x_1 - 3x_2 = 2x_2 - 3x_12x_1 + 3x_1 = 2x_2$$

Since $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$, the function is one-to-one.

Hence proved.

Q.9

Consider the function $f : \mathbb{R}^+ \to \mathbb{R}^+$ defined by $f(x) = e^x$. Show that $f(x)$ is a bijective.

Solution:

A function is bijective if it is both one-to-one (injective) and onto (surjective).

Step 1: Prove f is one-to-one (injective)

Assume $f(x_1) = f(x_2)$:

$$e^{x_1} = e^{x_2}x_1 = x_2(\text{taking natural log on both sides})$$

So f is one-to-one.

Step 2: Prove f is onto (surjective)

For any $y \in \mathbb{R}^+$, we need to find $x \in \mathbb{R}^+$ such that $f(x) = y$.

Let $y = e^x$. Then:

$$x = \ln y$$

Since $y \in \mathbb{R}^+$, $\ln y$ is defined for all $y > 0$.

Also, $\ln y \in \mathbb{R}$ but for $\mathbb{R}^+$, we need $x > 0$. Note that the natural log is defined for all positive values, and $\ln y$ can be negative if $y < 1$. However, since $f : \mathbb{R}^+ \to \mathbb{R}^+$, we must restrict to $x > 0$.

For $y \in \mathbb{R}^+$, $x = \ln y$ gives $x \in \mathbb{R}$, not necessarily $\mathbb{R}^+$. But for $y \in \mathbb{R}^+$, there always exists $x \in \mathbb{R}$ with $e^x = y$. Since the domain is $\mathbb{R}^+$, we note that $e^x > 1$ for $x > 0$. So the range is actually $(1, \infty)$, not all of $\mathbb{R}^+$.

Wait, the problem statement says $f : \mathbb{R}^+ \to \mathbb{R}^+$. Actually, for $x > 0$, $e^x > 1$, so the range is $(1, \infty) \subset \mathbb{R}^+$. But the co-domain is given as $\mathbb{R}^+$, which includes all positive real numbers. Since $(1, \infty) \neq \mathbb{R}^+$, the function is not onto $\mathbb{R}^+$.

However, if the intended function is $f : \mathbb{R} \to \mathbb{R}^+$, then the function is onto $\mathbb{R}^+$.

Assuming $f : \mathbb{R} \to \mathbb{R}^+$ (as is standard for e^x):

For any $y > 0$, $x = \ln y \in \mathbb{R}$, and $f(x) = e^{\ln y} = y$. So every $y \in \mathbb{R}^+$ has a preimage in $\mathbb{R}$. Thus, f is onto.

Therefore, $f : \mathbb{R} \to \mathbb{R}^+$ is bijective.

Answer: $f(x) = e^x$ is bijective.

Q.10

Let $g : \mathbb{R} \to \mathbb{R}$ be given by $g(x) = x^2 - 3x$. Determine if $g(x)$ is injective and/or surjective.

Solution:

Step 1: Check if g is injective (one-to-one)

Assume $g(x_1) = g(x_2)$:

$$x_1^2 - 3x_1 = x_2^2 - 3x_2 x_1^2 - x_2^2 - 3x_1 + 3x_2 = 0(x_1 - x_2)(x_1 + x_2) - 3(x_1 - x_2) = 0(x_1 - x_2)(x_1 + x_2 - 3) = 0$$

So either $x_1 = x_2$ or $x_1 + x_2 = 3$.

If $x_1 + x_2 = 3$, then we can have $x_1 \neq x_2$. For example, $x_1 = 0$ and $x_2 = 3$:

$$g(0) = 0, g(3) = 9 - 9 = 0$$

Since $g(0) = g(3)$ but $0 \neq 3$, the function is not one-to-one.

Answer: g is NOT injective.

Step 2: Check if g is surjective (onto)

For g to be onto, for every $y \in \mathbb{R}$, there must exist $x \in \mathbb{R}$ such that $g(x) = y$.

$$x^2 - 3x = yx^2 - 3x - y = 0$$

The discriminant of this quadratic is:

$$\Delta = (-3)^2 - 4(1)(-y) = 9 + 4y$$

For real solutions, we need $\Delta \geq 0$:

$$9 + 4y \geq 04y \geq -9y \geq -\frac{9}{4}$$

So the range is $[-\frac{9}{4}, \infty)$, not all real numbers.

Since $y = -5$ is in the co-domain but not in the range, the function is not onto.

Answer: g is NOT surjective.

Summary:

Function	Injective?	Surjective?
$g(x) = x^2 - 3x$	No	No