# EXERCISE 4.3 - COMPLETE SOLUTIONS

## Question 1: Find the inverses of the following matrices by using row operations

### (i) $\mathbf{A} = \begin{bmatrix} \mathbf{2} & \mathbf{6} & \mathbf{-3} \\ \mathbf{0} & \mathbf{-2} & \mathbf{0} \\ \mathbf{-2} & \mathbf{5} & \mathbf{6} \end{bmatrix}$

**Step 1:** Write the augmented matrix $[A \mid I]$

$$\begin{bmatrix} 2 & 6 & -3 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 & 1 & 0 \\ -2 & 5 & 6 & 0 & 0 & 1 \end{bmatrix}$$

**Step 2:** $R_3 \to R_3 + R_1$

$$\begin{bmatrix} 2 & 6 & -3 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 & 1 & 0 \\ 0 & 11 & 3 & 1 & 0 & 1 \end{bmatrix}$$

**Step 3:** $R_1 \to \frac{1}{2}R_1$, $R_2 \to -\frac{1}{2}R_2$

$$\begin{bmatrix} 1 & 3 & -\frac{3}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 11 & 3 & 1 & 0 & 1 \end{bmatrix}$$

**Step 4:** $R_3 \to R_3 - 11R_2$

$$\begin{bmatrix} 1 & 3 & -\frac{3}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 3 & 1 & \frac{11}{2} & 1 \end{bmatrix}$$

**Step 5:** $R_3 \to \frac{1}{3}R_3$

$$\begin{bmatrix} 1 & 3 & -\frac{3}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{1}{3} & \frac{11}{6} & \frac{1}{3} \end{bmatrix}$$

**Step 6:** $R_1 \to R_1 - 3R_2 + \frac{3}{2}R_3$

$$\begin{bmatrix} 1 & 0 & 0 & \frac{1}{2} + \frac{1}{2} & \frac{3}{2} + \frac{11}{4} & \frac{1}{2} \\ 0 & 1 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{1}{3} & \frac{11}{6} & \frac{1}{3} \end{bmatrix}$$

**Step 7:** Simplify

$$A^{-1} = \begin{bmatrix} 1 & \frac{17}{4} & \frac{1}{2} \\ 0 & -\frac{1}{2} & 0 \\ \frac{1}{3} & \frac{11}{6} & \frac{1}{3} \end{bmatrix}$$

**Answer:** $A^{-1} = \begin{bmatrix} 1 & \frac{17}{4} & \frac{1}{2} \\ 0 & -\frac{1}{2} & 0 \\ \frac{1}{3} & \frac{11}{6} & \frac{1}{3} \end{bmatrix}$



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**(ii)** $\mathbf{A} = \begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{-1} \\ \mathbf{0} & \mathbf{-2} & \mathbf{8} \\ \mathbf{1} & \mathbf{0} & \mathbf{2} \end{bmatrix}$

**Step 1:** Write the augmented matrix $[A \mid I]$

$$\begin{bmatrix} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -2 & 8 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{bmatrix}$$

**Step 2:** $R_3 \to R_3 - R_1$

$$\begin{bmatrix} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -2 & 8 & 0 & 1 & 0 \\ 0 & -2 & 3 & -1 & 0 & 1 \end{bmatrix}$$

**Step 3:** $R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -2 & 8 & 0 & 1 & 0 \\ 0 & 0 & -5 & -1 & -1 & 1 \end{bmatrix}$$

**Step 4:** $R_2 \to -\frac{1}{2}R_2$, $R_3 \to -\frac{1}{5}R_3$

$$\begin{bmatrix} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -4 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{1}{5} & \frac{1}{5} & -\frac{1}{5} \end{bmatrix}$$

**Step 5:** $R_1 \to R_1 - 2R_2 + R_3$

$$\begin{bmatrix} 1 & 0 & 0 & 1+0+\frac{1}{5} & 0+1-\frac{1}{5} & 0+0-\frac{1}{5} \\ 0 & 1 & -4 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{1}{5} & \frac{1}{5} & -\frac{1}{5} \end{bmatrix}$$

**Step 6:** $R_2 \to R_2 + 4R_3$

$$A^{-1} = \begin{bmatrix} \frac{6}{5} & \frac{4}{5} & -\frac{1}{5} \\ \frac{4}{5} & \frac{3}{10} & -\frac{4}{5} \\ \frac{1}{5} & \frac{1}{5} & -\frac{1}{5} \end{bmatrix}$$

**Answer:** $A^{-1} = \begin{bmatrix} \frac{6}{5} & \frac{4}{5} & -\frac{1}{5} \\ \frac{4}{5} & \frac{3}{10} & -\frac{4}{5} \\ \frac{1}{5} & \frac{1}{5} & -\frac{1}{5} \end{bmatrix}$

**(iii)** $\mathbf{A} = \begin{bmatrix} \mathbf{1} & \mathbf{6} & \mathbf{2} \\ \mathbf{2} & \mathbf{13} & \mathbf{0} \\ \mathbf{0} & \mathbf{-1} & \mathbf{1} \end{bmatrix}$

**Step 1:** Write the augmented matrix $[A \mid I]$

$$\begin{bmatrix} 1 & 6 & 2 & 1 & 0 & 0 \\ 2 & 13 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{bmatrix}$$



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**Step 2:** $R_2 \to R_2 - 2R_1$

$$\begin{bmatrix} 1 & 6 & 2 & 1 & 0 & 0 \\ 0 & 1 & -4 & -2 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

**Step 3:** $R_3 \to R_3 + R_2$

$$\begin{bmatrix} 1 & 6 & 2 & 1 & 0 & 0 \\ 0 & 1 & -4 & -2 & 1 & 0 \\ 0 & 0 & -3 & -2 & 1 & 1 \end{bmatrix}$$

**Step 4:** $R_3 \to -\frac{1}{3}R_3$

$$\begin{bmatrix} 1 & 6 & 2 & 1 & 0 & 0 \\ 0 & 1 & -4 & -2 & 1 & 0 \\ 0 & 0 & 1 & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \end{bmatrix}$$

**Step 5:** $R_2 \to R_2 + 4R_3$

$$\begin{bmatrix} 1 & 6 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -\frac{2}{3} & -\frac{1}{3} & -\frac{4}{3} \\ 0 & 0 & 1 & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \end{bmatrix}$$

**Step 6:** $R_1 \to R_1 - 6R_2 - 2R_3$

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 & 13 & 2 & 6 \\ 0 & 1 & 0 & -\frac{2}{3} & -\frac{1}{3} & -\frac{4}{3} \\ 0 & 0 & 1 & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \end{bmatrix}$$

**Answer:** $A^{-1} = \begin{bmatrix} 13 & 2 & 6 \\ -\frac{2}{3} & -\frac{1}{3} & -\frac{4}{3} \\ \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \end{bmatrix}$

## Question 2: Find the rank of the following matrices

### (i) $\mathbf{A} = \begin{bmatrix} \mathbf{1} & \mathbf{-1} & \mathbf{3} & \mathbf{1} \\ \mathbf{-2} & \mathbf{-6} & \mathbf{1} & \mathbf{-1} \\ \mathbf{3} & \mathbf{1} & \mathbf{4} & \mathbf{-2} \end{bmatrix}$

**Step 1:** Write the matrix

$$\begin{bmatrix} 1 & -1 & 3 & 1 \\ -2 & -6 & 1 & -1 \\ 3 & 1 & 4 & -2 \end{bmatrix}$$

**Step 2:** $R_2 \to R_2 + 2R_1$, $R_3 \to R_3 - 3R_1$

$$\begin{bmatrix} 1 & -1 & 3 & 1 \\ 0 & -8 & 7 & 1 \\ 0 & 4 & -5 & -5 \end{bmatrix}$$

**Step 3:** $R_3 \to R_3 + \frac{1}{2}R_2$



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$$\begin{bmatrix} 1 & -1 & 3 & 1 \\ 0 & -8 & 7 & 1 \\ 0 & 0 & -\frac{3}{2} & -\frac{9}{2} \end{bmatrix}$$

**Step 4:** The matrix has 3 non-zero rows (row echelon form has 3 pivots)

**Answer:** Rank = 3

**(ii)** $\mathbf{A} = \begin{bmatrix} \mathbf{1} & \mathbf{-2} & \mathbf{3} \\ \mathbf{2} & \mathbf{-4} & \mathbf{6} \\ \mathbf{-1} & \mathbf{0} & \mathbf{2} \\ \mathbf{0} & \mathbf{1} & \mathbf{-1} \end{bmatrix}$

**Step 1:** Write the matrix

$$\begin{bmatrix} 1 & -2 & 3 \\ 2 & -4 & 6 \\ -1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix}$$

**Step 2:** $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 + R_1$

$$\begin{bmatrix} 1 & -2 & 3 \\ 0 & 0 & 0 \\ 0 & -2 & 5 \\ 0 & 1 & -1 \end{bmatrix}$$

**Step 3:** Swap $R_3 \leftrightarrow R_4$

$$\begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & -1 \\ 0 & -2 & 5 \\ 0 & 0 & 0 \end{bmatrix}$$

**Step 4:** $R_3 \to R_3 + 2R_2$

$$\begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

**Step 5:** The matrix has 3 non-zero rows

**Answer:** Rank = 3

**(iii)** $\mathbf{A} = \begin{bmatrix} \mathbf{3} & \mathbf{-1} & \mathbf{3} & \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{2} & \mathbf{-1} & \mathbf{-3} & \mathbf{-2} \\ \mathbf{2} & \mathbf{3} & \mathbf{4} & \mathbf{2} & \mathbf{1} \\ \mathbf{2} & \mathbf{5} & \mathbf{-2} & \mathbf{-3} & \mathbf{3} \end{bmatrix}$

**Step 1:** $R_1 \leftrightarrow R_2$



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$$\begin{bmatrix} 1 & 2 & -1 & -3 & -2 \\ 3 & -1 & 3 & 0 & 1 \\ 2 & 3 & 4 & 2 & 1 \\ 2 & 5 & -2 & -3 & 3 \end{bmatrix}$$

**Step 2:** $R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - 2R_1$, $R_4 \to R_4 - 2R_1$

$$\begin{bmatrix} 1 & 2 & -1 & -3 & -2 \\ 0 & -7 & 6 & 9 & 7 \\ 0 & -1 & 6 & 8 & 5 \\ 0 & 1 & 0 & 3 & 7 \end{bmatrix}$$

**Step 3:** Swap $R_2 \leftrightarrow R_4$

$$\begin{bmatrix} 1 & 2 & -1 & -3 & -2 \\ 0 & 1 & 0 & 3 & 7 \\ 0 & -1 & 6 & 8 & 5 \\ 0 & -7 & 6 & 9 & 7 \end{bmatrix}$$

**Step 4:** $R_3 \to R_3 + R_2$, $R_4 \to R_4 + 7R_2$

$$\begin{bmatrix} 1 & 2 & -1 & -3 & -2 \\ 0 & 1 & 0 & 3 & 7 \\ 0 & 0 & 6 & 11 & 12 \\ 0 & 0 & 6 & 30 & 56 \end{bmatrix}$$

**Step 5:** $R_4 \to R_4 - R_3$

$$\begin{bmatrix} 1 & 2 & -1 & -3 & -2 \\ 0 & 1 & 0 & 3 & 7 \\ 0 & 0 & 6 & 11 & 12 \\ 0 & 0 & 0 & 19 & 44 \end{bmatrix}$$

**Step 6:** The matrix has 4 non-zero rows

**Answer:** Rank = 4

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## Question 3: Solve the following systems of linear equations by Cramer's rule

### (i) $\mathbf{x - y + z = 3,\ 2x + y + z = 7,\ x - 2y - z = -1}$

**Step 1:** Write in matrix form $AX = B$

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & 1 \\ 1 & -2 & -1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 3 \\ 7 \\ -1 \end{bmatrix}$$

**Step 2:** Find $|A|$

- $|A| = 1 \begin{vmatrix} 1 & 1 \\ -2 & -1 \end{vmatrix} - (-1) \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} + 1 \begin{vmatrix} 2 & 1 \\ 1 & -2 \end{vmatrix}$
- $= 1(-1 + 2) + 1(-2 - 1) + 1(-4 - 1)$
- $= 1(1) + 1(-3) + 1(-5) = 1 - 3 - 5 = -7$

**Step 3:** Find $|A_x|$ (replace column 1 with B)

- $A_x = \begin{bmatrix} 3 & -1 & 1 \\ 7 & 1 & 1 \\ -1 & -2 & -1 \end{bmatrix}$
- $|A_x| = 3 \begin{vmatrix} 1 & 1 \\ -2 & -1 \end{vmatrix} - (-1) \begin{vmatrix} 7 & 1 \\ -1 & -1 \end{vmatrix} + 1 \begin{vmatrix} 7 & 1 \\ -1 & -2 \end{vmatrix}$
- $= 3(-1+2) + 1(-7+1) + 1(-14+1)$
- $= 3(1) + 1(-6) + 1(-13) = 3 - 6 - 13 = -16$

**Step 4:** Find $|A_y|$ (replace column 2 with B)

- $A_y = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 7 & 1 \\ 1 & -1 & -1 \end{bmatrix}$
- $|A_y| = 1 \begin{vmatrix} 7 & 1 \\ -1 & -1 \end{vmatrix} - 3 \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} + 1 \begin{vmatrix} 2 & 7 \\ 1 & -1 \end{vmatrix}$
- $= 1(-7+1) - 3(-2-1) + 1(-2-7)$
- $= 1(-6) - 3(-3) + 1(-9) = -6 + 9 - 9 = -6$

**Step 5:** Find $|A_z|$ (replace column 3 with B)

- $A_z = \begin{bmatrix} 1 & -1 & 3 \\ 2 & 1 & 7 \\ 1 & -2 & -1 \end{bmatrix}$
- $|A_z| = 1 \begin{vmatrix} 1 & 7 \\ -2 & -1 \end{vmatrix} - (-1) \begin{vmatrix} 2 & 7 \\ 1 & -1 \end{vmatrix} + 3 \begin{vmatrix} 2 & 1 \\ 1 & -2 \end{vmatrix}$
- $= 1(-1+14) + 1(-2-7) + 3(-4-1)$
- $= 1(13) + 1(-9) + 3(-5) = 13 - 9 - 15 = -11$

**Step 6:** Apply Cramer's rule

- $x = \frac{|A_x|}{|A|} = \frac{-16}{-7} = \frac{16}{7}$
- $y = \frac{|A_y|}{|A|} = \frac{-6}{-7} = \frac{6}{7}$
- $z = \frac{|A_z|}{|A|} = \frac{-11}{-7} = \frac{11}{7}$

**Answer:** $x = \frac{16}{7}, y = \frac{6}{7}, z = \frac{11}{7}$

**(ii) $\mathbf{4x_1 - x_2 + x_3 = 5,\ -2x_1 + 3x_2 + 2x_3 = 3,\ x_1 - 2x_2 - x_3 = 1}$**

**Step 1:** Matrix form

$$A = \begin{bmatrix} 4 & -1 & 1 \\ -2 & 3 & 2 \\ 1 & -2 & -1 \end{bmatrix}, B = \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix}$$

**Step 2:** Find $|A|$

- $|A| = 4 \begin{vmatrix} 3 & 2 \\ -2 & -1 \end{vmatrix} - (-1) \begin{vmatrix} -2 & 2 \\ 1 & -1 \end{vmatrix} + 1 \begin{vmatrix} -2 & 3 \\ 1 & -2 \end{vmatrix}$
- $= 4(-3+4) + 1(2-2) + 1(4-3)$

- $= 4(1) + 1(0) + 1(1) = 4 + 0 + 1 = 5$

**Step 3:** Find $|A_{x1}|$

- $A_{x1} = \begin{bmatrix} 5 & -1 & 1 \\ 3 & 3 & 2 \\ 1 & -2 & -1 \end{bmatrix}$
- $|A_{x1}| = 5 \begin{vmatrix} 3 & 2 \\ -2 & -1 \end{vmatrix} - (-1) \begin{vmatrix} 3 & 2 \\ 1 & -1 \end{vmatrix} + 1 \begin{vmatrix} 3 & 3 \\ 1 & -2 \end{vmatrix}$
- $= 5(-3+4) + 1(-3-2) + 1(-6-3)$
- $= 5(1) + 1(-5) + 1(-9) = 5 - 5 - 9 = -9$

**Step 4:** Find $|A_{x2}|$

- $A_{x2} = \begin{bmatrix} 4 & 5 & 1 \\ -2 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix}$
- $|A_{x2}| = 4 \begin{vmatrix} 3 & 2 \\ 1 & -1 \end{vmatrix} - 5 \begin{vmatrix} -2 & 2 \\ 1 & -1 \end{vmatrix} + 1 \begin{vmatrix} -2 & 3 \\ 1 & 1 \end{vmatrix}$
- $= 4(-3-2) - 5(2-2) + 1(-2-3)$
- $= 4(-5) - 5(0) + 1(-5) = -20 - 0 - 5 = -25$

**Step 5:** Find $|A_{x3}|$

- $A_{x3} = \begin{bmatrix} 4 & -1 & 5 \\ -2 & 3 & 3 \\ 1 & -2 & 1 \end{bmatrix}$
- $|A_{x3}| = 4 \begin{vmatrix} 3 & 3 \\ -2 & 1 \end{vmatrix} - (-1) \begin{vmatrix} -2 & 3 \\ 1 & 1 \end{vmatrix} + 5 \begin{vmatrix} -2 & 3 \\ 1 & -2 \end{vmatrix}$
- $= 4(3+6) + 1(-2-3) + 5(4-3)$
- $= 4(9) + 1(-5) + 5(1) = 36 - 5 + 5 = 36$

**Step 6:** Apply Cramer's rule

- $x_1 = \frac{-9}{5}$, $x_2 = \frac{-25}{5} = -5$, $x_3 = \frac{36}{5}$

**Answer:** $x_1 = -\frac{9}{5}, x_2 = -5, x_3 = \frac{36}{5}$

### (iii) $\mathbf{x_1 + 2x_2 + 2x_3 = 2,\ 3x_1 + 2y + z = 4,\ x_1 - 2x_2 - x_3 = 1}$

**Note:** The second equation has $3x_1 + 2y + z = 4$. Assuming $y = x_2$ and $z = x_3$:

**Step 1:** Matrix form

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 3 & 2 & 1 \\ 1 & -2 & -1 \end{bmatrix}, B = \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}$$

**Step 2:** Find $|A|$

- $|A| = 1 \begin{vmatrix} 2 & 1 \\ -2 & -1 \end{vmatrix} - 2 \begin{vmatrix} 3 & 1 \\ 1 & -1 \end{vmatrix} + 2 \begin{vmatrix} 3 & 2 \\ 1 & -2 \end{vmatrix}$
- $= 1(-2+2) - 2(-3-1) + 2(-6-2)$
- $= 1(0) - 2(-4) + 2(-8) = 0 + 8 - 16 = -8$



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**Step 3:** Find $|A_{x1}|$

- $A_{x1} = \begin{bmatrix} 2 & 2 & 2 \\ 4 & 2 & 1 \\ 1 & -2 & -1 \end{bmatrix}$
- $|A_{x1}| = 2\begin{vmatrix} 2 & 1 \\ -2 & -1 \end{vmatrix} - 2\begin{vmatrix} 4 & 1 \\ 1 & -1 \end{vmatrix} + 2\begin{vmatrix} 4 & 2 \\ 1 & -2 \end{vmatrix}$
- $= 2(-2+2) - 2(-4-1) + 2(-8-2)$
- $= 2(0) - 2(-5) + 2(-10) = 0 + 10 - 20 = -10$

**Step 4:** Find $|A_{x2}|$

- $A_{x2} = \begin{bmatrix} 1 & 2 & 2 \\ 3 & 4 & 1 \\ 1 & 1 & -1 \end{bmatrix}$
- $|A_{x2}| = 1\begin{vmatrix} 4 & 1 \\ 1 & -1 \end{vmatrix} - 2\begin{vmatrix} 3 & 1 \\ 1 & -1 \end{vmatrix} + 2\begin{vmatrix} 3 & 4 \\ 1 & 1 \end{vmatrix}$
- $= 1(-4-1) - 2(-3-1) + 2(3-4)$
- $= 1(-5) - 2(-4) + 2(-1) = -5 + 8 - 2 = 1$

**Step 5:** Find $|A_{x3}|$

- $A_{x3} = \begin{bmatrix} 1 & 2 & 2 \\ 3 & 2 & 4 \\ 1 & -2 & 1 \end{bmatrix}$
- $|A_{x3}| = 1\begin{vmatrix} 2 & 4 \\ -2 & 1 \end{vmatrix} - 2\begin{vmatrix} 3 & 4 \\ 1 & 1 \end{vmatrix} + 2\begin{vmatrix} 3 & 2 \\ 1 & -2 \end{vmatrix}$
- $= 1(2+8) - 2(3-4) + 2(-6-2)$
- $= 1(10) - 2(-1) + 2(-8) = 10 + 2 - 16 = -4$

**Step 6:** Apply Cramer's rule

- $x_1 = \frac{-10}{-8} = \frac{5}{4}$, $x_2 = \frac{1}{-8} = -\frac{1}{8}$, $x_3 = \frac{-4}{-8} = \frac{1}{2}$

**Answer:** $x_1 = \frac{5}{4}, x_2 = -\frac{1}{8}, x_3 = \frac{1}{2}$

### Question 4: Solve by matrix inversion method

#### (i) $\mathbf{3x + y - 2z = 4,\ x_1 + 3x_2 - 2x_3 = 0,\ 2x - z = 1,\ y - z = 1}$

**Note:** The system has 3 equations but 4 variables. Let's assume:

- $3x + y - 2z = 4$
- $2x - z = 1$
- $y - z = 1$

**Step 1:** From $2x - z = 1$, we get $z = 2x - 1$
**Step 2:** From $y - z = 1$, we get $y = z + 1 = 2x - 1 + 1 = 2x$
**Step 3:** Substitute in $3x + y - 2z = 4$:

- $3x + 2x - 2(2x - 1) = 4$
- $5x - 4x + 2 = 4$
- $x + 2 = 4$



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- $x = 2$

**Step 4:** Find y and z

- $y = 2x = 4$
- $z = 2x - 1 = 4 - 1 = 3$

**Answer:** $x = 2, y = 4, z = 3$

### (ii) $-3x_1 - x_2 + 2x_3 = 4$, $x_1 + 3x_2 - 2x_3 = 0$, $2y - 3z = -1$

**Note:** This has mixed variables. Let's solve:

- $-3x_1 - x_2 + 2x_3 = 4$ ... (1)
- $x_1 + 3x_2 - 2x_3 = 0$ ... (2)
- $2x_2 - 3x_3 = -1$ ... (3) (assuming y = x , z = x )

**Step 1:** Add (1) and (2):

- $-3x_1 + x_1 - x_2 + 3x_2 + 2x_3 - 2x_3 = 4 + 0$
- $-2x_1 + 2x_2 = 4$
- $-x_1 + x_2 = 2$ ... (4)

**Step 2:** From (3): $2x_2 - 3x_3 = -1$ ... (3)

**Step 3:** From (2): $x_1 = -3x_2 + 2x_3$

**Step 4:** Substitute in (4):

- $-(-3x_2 + 2x_3) + x_2 = 2$
- $3x_2 - 2x_3 + x_2 = 2$
- $4x_2 - 2x_3 = 2$
- $2x_2 - x_3 = 1$ ... (5)

**Step 5:** Solve (3) and (5):

- From (5): $x_3 = 2x_2 - 1$
- Substitute in (3): $2x_2 - 3(2x_2 - 1) = -1$
- $2x_2 - 6x_2 + 3 = -1$
- $-4x_2 = -4$
- $x_2 = 1$

**Step 6:** Find $x_3$ and $x_1$

- $x_3 = 2(1) - 1 = 1$
- $x_1 = -3(1) + 2(1) = -1$

**Answer:** $x_1 = -1, x_2 = 1, x_3 = 1$

## Question 5: Solve by reducing augmented matrices to echelon form

### (i) $x_1 + 2x_2 - 2x_3 = -1$, $2x_1 + 3x_2 + x_3 = 1$, $2x + y + z = 2$

**Note:** Assuming variables are consistent ($x = x_1, y = x_2, z = x_3$):

$$\begin{bmatrix} 1 & 2 & -2 & | & -1 \\ 2 & 3 & 1 & | & 1 \\ 2 & 1 & 1 & | & 2 \end{bmatrix}$$

**Step 1:** $R_2 \to R_2 - 2R_1,\ R_3 \to R_3 - 2R_1$

$$\begin{bmatrix} 1 & 2 & -2 & | & -1 \\ 0 & -1 & 5 & | & 3 \\ 0 & -3 & 5 & | & 4 \end{bmatrix}$$

**Step 2:** $R_3 \to R_3 - 3R_2$

$$\begin{bmatrix} 1 & 2 & -2 & | & -1 \\ 0 & -1 & 5 & | & 3 \\ 0 & 0 & -10 & | & -5 \end{bmatrix}$$

**Step 3:** Back substitution:

- From $R_3$: $-10z = -5 \to z = \frac{1}{2}$
- From $R_2$: $-y + 5z = 3 \to -y + \frac{5}{2} = 3 \to y = -\frac{1}{2}$
- From $R_1$: $x + 2y - 2z = -1 \to x - 1 - 1 = -1 \to x = 1$

**Answer:** $x = 1, y = -\frac{1}{2}, z = \frac{1}{2}$

### (ii) $\mathbf{2x + y + 2z = 3,\ x + y - z = 1,\ 2x_1 + x_2 - 2x_3 = 9,\ 3x_1 + x_2 - x_3 = 12}$

**Step 1:** Write the augmented matrix for the last two equations:

$$\begin{bmatrix} 2 & 1 & -2 & | & 9 \\ 3 & 1 & -1 & | & 12 \end{bmatrix}$$

**Step 2:** $R_2 \to R_2 - \frac{3}{2}R_1$

$$\begin{bmatrix} 2 & 1 & -2 & | & 9 \\ 0 & -\frac{1}{2} & 2 & | & -\frac{3}{2} \end{bmatrix}$$

**Step 3:** Back substitution:

- From $R_2$: $-\frac{1}{2}x_2 + 2x_3 = -\frac{3}{2} \to x_2 = 4x_3 + 3$
- From $R_1$: $2x_1 + x_2 - 2x_3 = 9$
- $2x_1 + (4x_3 + 3) - 2x_3 = 9$
- $2x_1 + 2x_3 + 3 = 9$
- $x_1 + x_3 = 3$
- $x_1 = 3 - x_3$

**Answer:** $x_1 = 3 - t, x_2 = 3 + 4t, x_3 = t$ (infinite solutions)

**Summary Table**

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| | |
|---|---|
| 1(i) | $\begin{bmatrix} 1 & \frac{17}{4} & \frac{1}{2} \\ 0 & -\frac{1}{2} & 0 \\ \frac{1}{3} & \frac{11}{6} & \frac{1}{3} \end{bmatrix}$ |
| 1(ii) | $\begin{bmatrix} \frac{6}{5} & \frac{4}{5} & -\frac{1}{5} \\ \frac{4}{5} & \frac{3}{10} & -\frac{4}{5} \\ \frac{1}{5} & \frac{1}{5} & -\frac{1}{5} \end{bmatrix}$ |
| 1(iii) | $\begin{bmatrix} 13 & 2 & 6 \\ -\frac{2}{3} & -\frac{1}{3} & -\frac{4}{3} \\ \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \end{bmatrix}$ |
| 2(i) | Rank = 3 |
| 2(ii) | Rank = 3 |
| 2(iii) | Rank = 4 |
| 3(i) | $x = \frac{16}{7}, y = \frac{6}{7}, z = \frac{11}{7}$ |
| 3(ii) | $x_1 = -\frac{9}{5}, x_2 = -5, x_3 = \frac{36}{5}$ |
| 3(iii) | $x_1 = \frac{5}{4}, x_2 = -\frac{1}{8}, x_3 = \frac{1}{2}$ |
| 4(i) | $x = 2, y = 4, z = 3$ |
| 4(ii) | $x_1 = -1, x_2 = 1, x_3 = 1$ |
| 5(i) | $x = 1, y = -\frac{1}{2}, z = \frac{1}{2}$ |
| 5(ii) | $x_1 = 3 - t, x_2 = 3 + 4t, x_3 = t$ |

**EXERCISE 4.3 - COMPLETE SOLUTIONS (Continued)**

### Question 6: Solve homogeneous systems by Gaussian elimination

**(i)** $\begin{cases} \mathbf{x + 4y - 2z = 0} \\ \mathbf{2x + y + 5z = 0} \\ \mathbf{5x + 2y + 8z = 0} \end{cases}$

**Step 1:** Write the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 4 & -2 & 0 \\ 2 & 1 & 5 & 0 \\ 5 & 2 & 8 & 0 \end{array}\right]$$

**Step 2:** $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 5R_1$

$$\left[\begin{array}{ccc|c} 1 & 4 & -2 & 0 \\ 0 & -7 & 9 & 0 \\ 0 & -18 & 18 & 0 \end{array}\right]$$

**Step 3:** $R_3 \to R_3 - \frac{18}{7}R_2$

$$\left[\begin{array}{ccc|c} 1 & 4 & -2 & 0 \\ 0 & -7 & 9 & 0 \\ 0 & 0 & -\frac{36}{7} & 0 \end{array}\right]$$

**Step 4:** Back substitution

- From $R_3$: $-\frac{36}{7}z = 0 \to z = 0$
- From $R_2$: $-7y + 9(0) = 0 \to y = 0$
- From $R_1$: $x + 4(0) - 2(0) = 0 \to x = 0$



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**Answer:** Only trivial solution: $x = 0, y = 0, z = 0$

**(ii)** $\begin{cases} \mathbf{x_1+4x_2+2x_3=0} \\ \mathbf{2x_1+x_2-3x_3=0} \\ \mathbf{3x_1+2x_2-4x_3=0} \end{cases}$

**Step 1:** Write the augmented matrix

$$\begin{bmatrix} 1 & 4 & 2 & | & 0 \\ 2 & 1 & -3 & | & 0 \\ 3 & 2 & -4 & | & 0 \end{bmatrix}$$

**Step 2:** $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$

$$\begin{bmatrix} 1 & 4 & 2 & | & 0 \\ 0 & -7 & -7 & | & 0 \\ 0 & -10 & -10 & | & 0 \end{bmatrix}$$

**Step 3:** $R_3 \to R_3 - \frac{10}{7}R_2$

$$\begin{bmatrix} 1 & 4 & 2 & | & 0 \\ 0 & -7 & -7 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

**Step 4:** Back substitution

- From $R_2$: $-7x_2 - 7x_3 = 0 \to x_2 = -x_3$
- From $R_1$: $x_1 + 4x_2 + 2x_3 = 0 \to x_1 + 4(-x_3) + 2x_3 = 0 \to x_1 - 2x_3 = 0 \to x_1 = 2x_3$

**Answer:** $x_1 = 2t, x_2 = -t, x_3 = t$ (infinite solutions, where $t$ is arbitrary)

**(iii)** $\begin{cases} \mathbf{x_1+2x_2-x_3=0} \\ \mathbf{x_1-x_2+5x_3=0} \\ \mathbf{2x_1+x_2+4x_3=0} \end{cases}$

**Step 1:** Write the augmented matrix

$$\begin{bmatrix} 1 & 2 & -1 & | & 0 \\ 1 & -1 & 5 & | & 0 \\ 2 & 1 & 4 & | & 0 \end{bmatrix}$$

**Step 2:** $R_2 \to R_2 - R_1$, $R_3 \to R_3 - 2R_1$

$$\begin{bmatrix} 1 & 2 & -1 & | & 0 \\ 0 & -3 & 6 & | & 0 \\ 0 & -3 & 6 & | & 0 \end{bmatrix}$$

**Step 3:** $R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & 2 & -1 & | & 0 \\ 0 & -3 & 6 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

**Step 4:** Back substitution



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- From $R_2$: $-3x_2 + 6x_3 = 0 \rightarrow x_2 = 2x_3$
- From $R_1$: $x_1 + 2x_2 - x_3 = 0 \rightarrow x_1 + 4x_3 - x_3 = 0 \rightarrow x_1 = -3x_3$

**Answer:** $x_1 = -3t, x_2 = 2t, x_3 = t$ (infinite solutions, where $t$ is arbitrary)

### Question 7: Reflect triangle over y-axis

**Given:** $\mathbf{A(4,1),\ B(-2,5),\ C(0,-3)}$

**Step 1:** Reflection over y-axis transformation matrix

- $M = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

**Step 2:** Apply transformation to each vertex

- For $A(4,1)$: $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \end{bmatrix} \rightarrow A'(-4,1)$
- For $B(-2,5)$: $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix} \rightarrow B'(2,5)$
- For $C(0,-3)$: $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \end{bmatrix} \rightarrow C'(0,-3)$

**Answer:** $A'(-4,1)$, $B'(2,5)$, $C'(0,-3)$

### Question 8: Find coordinates of A

**Given:** $\mathbf{A' = (30, 20, -5)}$ **and scaling matrix**

$$P = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -5 \end{bmatrix}$$

**Step 1:** We know $PA = A'$. Let $A = (x, y, z)$

**Step 2:** Write the equation

$$\begin{bmatrix} 5 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 30 \\ 20 \\ -5 \end{bmatrix}$$

**Step 3:** Multiply:

- $5x = 30 \rightarrow x = 6$
- $-5y = 20 \rightarrow y = -4$
- $-5z = -5 \rightarrow z = 1$

**Answer:** $A = (6, -4, 1)$

### Question 9: Find equation of image of $\mathbf{y = x^2}$

**Given transformation matrix:**

$$M = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$



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**Step 1:** Write homogeneous coordinates for a point on $y = x^2$

- A point on the curve: $(x, y, 1)$ where $y = x^2$

**Step 2:** Apply the transformation:

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ x^2 \\ 1 \end{bmatrix}$$

**Step 3:** Multiply:

- $X = x + 1$
- $Y = 2x - x^2$
- $Z = 2$

**Step 4:** Convert to Cartesian coordinates:

- $x = X - 1$
- $y = \frac{Y}{Z} = \frac{2x-x^2}{2}$

**Step 5:** Substitute $x = X - 1$:

- $y = \frac{2(X-1)-(X-1)^2}{2}$
- $y = \frac{2X-2-(X^2-2X+1)}{2}$
- $y = \frac{2X-2-X^2+2X-1}{2}$
- $y = \frac{-X^2+4X-3}{2}$

**Answer:** $y = \frac{-x^2+4x-3}{2}$ or $2y = -x^2 + 4x - 3$

## Question 10: Encode message "KEEP IT UP" using matrix A

**Given:**

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Letters A-Z correspond to numbers 1-26. Space = 27.

**Step 1:** Convert message to numbers

- K = 11, E = 5, E = 5, P = 16, space = 27, I = 9, T = 20, space = 27, U = 21, P = 16

**Step 2:** Group into 3×1 column vectors (since A is 3×3)

- Group 1: K, E, E $\rightarrow \begin{bmatrix} 11 \\ 5 \\ 5 \end{bmatrix}$
- Group 2: P, space, I $\rightarrow \begin{bmatrix} 16 \\ 27 \\ 9 \end{bmatrix}$
- Group 3: T, space, U $\rightarrow \begin{bmatrix} 20 \\ 27 \\ 21 \end{bmatrix}$



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- Group 4: P, ?, ? $\rightarrow \begin{bmatrix} 16 \\ 0 \\ 0 \end{bmatrix}$ (padding with zeros)

**Step 3:** Multiply A by each vector

**For Group 1:**

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 11 \\ 5 \\ 5 \end{bmatrix} = \begin{bmatrix} 11+0+5 \\ 22-5+0 \\ 0+0+10 \end{bmatrix} = \begin{bmatrix} 16 \\ 17 \\ 10 \end{bmatrix}$$

**For Group 2:**

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 16 \\ 27 \\ 9 \end{bmatrix} = \begin{bmatrix} 16+0+9 \\ 32-27+0 \\ 0+0+18 \end{bmatrix} = \begin{bmatrix} 25 \\ 5 \\ 18 \end{bmatrix}$$

**For Group 3:**

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 20 \\ 27 \\ 21 \end{bmatrix} = \begin{bmatrix} 20+0+21 \\ 40-27+0 \\ 0+0+42 \end{bmatrix} = \begin{bmatrix} 41 \\ 13 \\ 42 \end{bmatrix}$$

**For Group 4:**

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 16 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 16 \\ 32 \\ 0 \end{bmatrix}$$

**Step 4:** Write encoded message as a matrix

$$\begin{bmatrix} 16 & 25 & 41 & 16 \\ 17 & 5 & 13 & 32 \\ 10 & 18 & 42 & 0 \end{bmatrix}$$

**Answer:** Encoded matrix $= \begin{bmatrix} 16 & 25 & 41 & 16 \\ 17 & 5 & 13 & 32 \\ 10 & 18 & 42 & 0 \end{bmatrix}$

## Question 11: Decode the message

**Given encoded matrix:**

$$C = \begin{bmatrix} 11 & 25 & 22 \\ 20 & 10 & 14 \\ 43 & 41 & 41 \end{bmatrix}$$

**Encoding matrix:**

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$



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**Step 1:** Find $A^{-1}$

**Step 1a:** Find $|\, A \,|$

- $|\, A \,| = 1 \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} - 1 \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} + (-1) \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix}$
- $= 1(0-1) - 1(1-2) - 1(1-0)$
- $= 1(-1) - 1(-1) - 1(1) = -1 + 1 - 1 = -1$

**Step 1b:** Find cofactor matrix

- $C_{11} = \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = -1$
- $C_{12} = -\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = -(1-2) = 1$
- $C_{13} = \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} = 1$
- $C_{21} = -\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = -(1+1) = -2$
- $C_{22} = \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} = 1 + 2 = 3$
- $C_{23} = -\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = -(1-2) = 1$
- $C_{31} = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1$
- $C_{32} = -\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = -(1+1) = -2$
- $C_{33} = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = -1$

**Step 1c:** Adjoint matrix (transpose of cofactor)

$$\text{adj}(A) = \begin{bmatrix} -1 & -2 & 1 \\ 1 & 3 & -2 \\ 1 & 1 & -1 \end{bmatrix}$$

**Step 1d:** $A^{-1} = \frac{1}{|A|}\text{adj}(A) = -1 \cdot \text{adj}(A)$

$$A^{-1} = \begin{bmatrix} 1 & 2 & -1 \\ -1 & -3 & 2 \\ -1 & -1 & 1 \end{bmatrix}$$

**Step 2:** Decode by multiplying $A^{-1}C$

**Column 1:** $\begin{bmatrix} 1 & 2 & -1 \\ -1 & -3 & 2 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 11 \\ 20 \\ 43 \end{bmatrix}$

- Row 1: $1(11) + 2(20) + (-1)(43) = 11 + 40 - 43 = 8$
- Row 2: $-1(11) - 3(20) + 2(43) = -11 - 60 + 86 = 15$

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- Row 3: $-1(11) - 1(20) + 1(43) = -11 - 20 + 43 = 12$

**Column 2:** $\begin{bmatrix} 1 & 2 & -1 \\ -1 & -3 & 2 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 25 \\ 10 \\ 41 \end{bmatrix}$

- Row 1: $25 + 20 - 41 = 4$
- Row 2: $-25 - 30 + 82 = 27$
- Row 3: $-25 - 10 + 41 = 6$

**Column 3:** $\begin{bmatrix} 1 & 2 & -1 \\ -1 & -3 & 2 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 22 \\ 14 \\ 41 \end{bmatrix}$

- Row 1: $22 + 28 - 41 = 9$
- Row 2: $-22 - 42 + 82 = 18$
- Row 3: $-22 - 14 + 41 = 5$

**Step 3:** Decoded matrix

$$\begin{bmatrix} 8 & 4 & 9 \\ 15 & 27 & 18 \\ 12 & 6 & 5 \end{bmatrix}$$

**Step 4:** Convert numbers to letters (1=A, ..., 26=Z, 27=space)

- $8 = $ H, $15 = $ O, $12 = $ L
- $4 = $ D, $27 = $ space, $6 = $ F
- $9 = $ I, $18 = $ R, $5 = $ E

**Step 5:** Read column by column:

- Column 1: 8=H, 15=O, 12=L $\rightarrow$ "HOL"
- Column 2: 4=D, 27=space, 6=F $\rightarrow$ "D F"
- Column 3: 9=I, 18=R, 5=E $\rightarrow$ "IRE"

Wait, this doesn't make sense. Let's read differently.

**Step 6:** Read row by row:

- Row 1: 8, 4, 9 $\rightarrow$ H, D, I $\rightarrow$ "HDI"
- Row 2: 15, 27, 18 $\rightarrow$ O, space, R $\rightarrow$ "O R"
- Row 3: 12, 6, 5 $\rightarrow$ L, F, E $\rightarrow$ "LFE"

This doesn't form meaningful text either. Let's check if we should read as 3-letter groups:

**Step 7:** Reading the decoded numbers in order (column by column):

- 8, 15, 12, 4, 27, 6, 9, 18, 5
- 8=H, 15=O, 12=L, 4=D, 27=space, 6=F, 9=I, 18=R, 5=E

**"HOLD FIRE"**

**Answer:** The decoded message is **"HOLD FIRE"**

**Summary Table**

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| | |
|---|---|
| 6(i) | $x = 0, y = 0, z = 0$ |
| 6(ii) | $x_1 = 2t, x_2 = -t, x_3 = t$ |
| 6(iii) | $x_1 = -3t, x_2 = 2t, x_3 = t$ |
| 7 | $A^{'}(-4, 1), B^{'}(2, 5), C^{'}(0, -3)$ |
| 8 | $A = (6, -4, 1)$ |
| 9 | $2y = -x^2 + 4x - 3$ |
| 10 | $\left.\begin{matrix} 16 & 25 & 41 & 16 \\ 17 & 5 & 13 & 32 \\ 10 & 18 & 42 & 0 \end{matrix}\right]$ |
| 11 | "HOLD FIRE" |

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