EXERCISE 5.1 - PARTIAL FRACTIONS

Complete Solutions

Question 1: $\frac{\mathbf{2}}{\mathbf{(x-a)(x-b)}}$

Step 1: Set up partial fraction decomposition

$$\frac{2}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}$$

Step 2: Multiply both sides by $(x-a)(x-b)$

$$2 = A(x-b) + B(x-a)$$

Step 3: Solve for A and B

Put $x = a$:

- $2 = A(a-b) + B(0)$
- $A = \frac{2}{a-b}$

Put $x = b$:

- $2 = A(0) + B(b-a)$
- $B = \frac{2}{b-a} = -\frac{2}{a-b}$

Answer: $\frac{2}{(x-a)(x-b)} = \frac{2}{a-b}\left(\frac{1}{x-a} - \frac{1}{x-b}\right)$

Question 2: $\frac{\mathbf{x^2+1}}{\mathbf{(x+1)(x-1)}}$

Step 1: Check if degree of numerator   degree of denominator

- Degree of numerator $= 2$, degree of denominator $= 2$
- Need to divide first

Step 2: Perform division

$$\frac{x^2+1}{(x+1)(x-1)} = \frac{x^2+1}{x^2-1} = 1 + \frac{2}{x^2-1}$$

Step 3: Decompose $\frac{2}{(x+1)(x-1)}$

$$\frac{2}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1}2 = A(x-1) + B(x+1)$$

Put $x = 1$: $2 = 2B \rightarrow B = 1$
Put $x = -1$: $2 = -2A \rightarrow A = -1$

Answer: $\frac{x^2+1}{(x+1)(x-1)} = 1 - \frac{1}{x+1} + \frac{1}{x-1}$

Question 3: $\frac{\mathbf{2x+3}}{\mathbf{x^2+4x+5}}$

Step 1: Check discriminant of denominator

- $x^2 + 4x + 5$ has discriminant $16 - 20 = -4 < 0$
- Irreducible quadratic factor

Step 2: Set up partial fraction

$$\frac{2x+3}{x^2+4x+5} = \frac{Ax+B}{x^2+4x+5}$$

Step 3: Compare coefficients

- $2x+3 = Ax+B$
- $A = 2$, $B = 3$

Answer: $\frac{2x+3}{x^2+4x+5} = \frac{2x+3}{x^2+4x+5}$ (already in simplest form)

Question 4: $\mathbf{\frac{4x^2+5x^2-3x-2}{(x+1)(x+2)(x+3)}}$

Note: $4x^2+5x^2 = 9x^2$, so numerator $= 9x^2-3x-2$

Step 1: Set up partial fraction decomposition

$$\frac{9x^2-3x-2}{(x+1)(x+2)(x+3)} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{x+3}$$

Step 2: Multiply both sides by $(x+1)(x+2)(x+3)$

$$9x^2-3x-2 = A(x+2)(x+3) + B(x+1)(x+3) + C(x+1)(x+2)$$

Step 3: Solve for A, B, C

Put $x = -1$:

- $9(-1)^2 - 3(-1) - 2 = 9+3-2 = 10$
- RHS: $A(-1+2)(-1+3) = A(1)(2) = 2A$
- $2A = 10 \to A = 5$

Put $x = -2$:

- $9(4) - 3(-2) - 2 = 36+6-2 = 40$
- RHS: $B(-2+1)(-2+3) = B(-1)(1) = -B$
- $-B = 40 \to B = -40$

Put $x = -3$:

- $9(9) - 3(-3) - 2 = 81+9-2 = 88$
- RHS: $C(-3+1)(-3+2) = C(-2)(-1) = 2C$
- $2C = 88 \to C = 44$

Answer: $\frac{9x^2-3x-2}{(x+1)(x+2)(x+3)} = \frac{5}{x+1} - \frac{40}{x+2} + \frac{44}{x+3}$

Question 5: $\mathbf{\frac{4}{(x+1)(x^2+5x+6)}}$

Step 1: Factor denominator

- $x^2+5x+6 = (x+2)(x+3)$
- Denominator $= (x+1)(x+2)(x+3)$

Step 2: Set up partial fraction

$$\frac{4}{(x+1)(x+2)(x+3)} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{x+3}$$

Step 3: Multiply both sides by $(x+1)(x+2)(x+3)$

$$4 = A(x+2)(x+3) + B(x+1)(x+3) + C(x+1)(x+2)$$

Step 4: Solve for A, B, C

Put $x = -1$:

- $4 = A(1)(2) = 2A \to A = 2$

Put $x = -2$:

- $4 = B(-1)(1) = -B \to B = -4$

Put $x = -3$:

- $4 = C(-2)(-1) = 2C \to C = 2$

Answer: $\frac{4}{(x+1)(x^2+5x+6)} = \frac{2}{x+1} - \frac{4}{x+2} + \frac{2}{x+3}$

Question 6: $\mathbf{\frac{5}{x^2-1}}$

Step 1: Factor denominator

- $x^2 - 1 = (x-1)(x+1)$

Step 2: Set up partial fraction

$$\frac{5}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}$$

Step 3: Multiply both sides by $(x-1)(x+1)$

$$5 = A(x+1) + B(x-1)$$

Step 4: Solve for A and B

Put $x = 1$:

- $5 = 2A \to A = \frac{5}{2}$

Put $x = -1$:

- $5 = -2B \to B = -\frac{5}{2}$

Answer: $\frac{5}{x^2-1} = \frac{5}{2}\left(\frac{1}{x-1} - \frac{1}{x+1}\right)$

Question 7: $\mathbf{\frac{3x^2-12x+11}{(x-1)(x-2)(x-3)}}$

Step 1: Set up partial fraction

$$\frac{3x^2-12x+11}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}$$

Step 2: Multiply both sides by $(x-1)(x-2)(x-3)$

$$3x^2 - 12x + 11 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)$$

Step 3: Solve for A, B, C

Put $x = 1$:

- $3(1) - 12(1) + 11 = 3 - 12 + 11 = 2$
- RHS: $A(-1)(-2) = 2A$
- $2A = 2 \to A = 1$

Put $x = 2$:

- $12 - 24 + 11 = -1$
- RHS: $B(1)(-1) = -B$
- $-B = -1 \to B = 1$

Put $x = 3$:

- $27 - 36 + 11 = 2$
- RHS: $C(2)(1) = 2C$
- $2C = 2 \to C = 1$

Answer: $\frac{3x^2-12x+11}{(x-1)(x-2)(x-3)} = \frac{1}{x-1} + \frac{1}{x-2} + \frac{1}{x-3}$

Question 8: $\mathbf{\frac{9}{(x-1)(x-2)(x-3)}}$

Step 1: Set up partial fraction

$$\frac{9}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}$$

Step 2: Multiply both sides by $(x-1)(x-2)(x-3)$

$$9 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)$$

Step 3: Solve for A, B, C

Put $x = 1$:

- $9 = A(-1)(-2) = 2A \to A = \frac{9}{2}$

Put $x = 2$:

- $9 = B(1)(-1) = -B \to B = -9$

Put $x = 3$:

- $9 = C(2)(1) = 2C \to C = \frac{9}{2}$

Answer: $\frac{9}{(x-1)(x-2)(x-3)} = \frac{9}{2(x-1)} - \frac{9}{x-2} + \frac{9}{2(x-3)}$

Question 9: $\mathbf{\frac{6}{(x-4)(x-5)(x-6)}}$

Step 1: Set up partial fraction

$$\frac{6}{(x-4)(x-5)(x-6)} = \frac{A}{x-4} + \frac{B}{x-5} + \frac{C}{x-6}$$

Step 2: Multiply both sides by $(x-4)(x-5)(x-6)$

$$6 = A(x-5)(x-6) + B(x-4)(x-6) + C(x-4)(x-5)$$

Step 3: Solve for A, B, C

Put $x = 4$:

- $6 = A(-1)(-2) = 2A \to A = 3$

Put $x = 5$:

- $6 = B(1)(-1) = -B \to B = -6$

Put $x = 6$:

- $6 = C(2)(1) = 2C \to C = 3$

Answer: $\frac{6}{(x-4)(x-5)(x-6)} = \frac{3}{x-4} - \frac{6}{x-5} + \frac{3}{x-6}$

Question 10: $\mathbf{\frac{4x^2+5x+1}{(x^2+a)(x^2+b)(x^2+c)}}$

Step 1: Note that denominator has irreducible quadratic factors

Step 2: Set up partial fraction

$$\frac{4x^2+5x+1}{(x^2+a)(x^2+b)(x^2+c)} = \frac{Ax+B}{x^2+a} + \frac{Cx+D}{x^2+b} + \frac{Ex+F}{x^2+c}$$

Step 3: Multiply both sides by $(x^2+a)(x^2+b)(x^2+c)$

$$4x^2+5x+1 = (Ax+B)(x^2+b)(x^2+c) + (Cx+D)(x^2+a)(x^2+c) + (Ex+F)(x^2+a)(x^2+b)$$

Step 4: This is a system of 6 equations with 6 unknowns. For specific values of a, b, c, we can solve.

Note: Without specific values for a, b, c, this is the general form.

Answer: General form shown above.

Question 11: $\mathbf{\frac{x+1}{(x-1)^2}}$

Step 1: Set up partial fraction (repeated linear factor)

$$\frac{x+1}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2}$$

Step 2: Multiply both sides by $(x-1)^2$

$$x+1 = A(x-1) + B$$

Step 3: Solve for A and B

- $x + 1 = Ax - A + B$
- Compare coefficients: $A = 1$, $-A + B = 1 \to -1 + B = 1 \to B = 2$

Answer: $\frac{x+1}{(x-1)^2} = \frac{1}{x-1} + \frac{2}{(x-1)^2}$

Question 12: $\mathbf{\frac{x^2+x}{(x^2-1)^2}}$

Step 1: Factor denominator

- $x^2 - 1 = (x-1)(x+1)$
- $(x^2-1)^2 = (x-1)^2{(x+1)}^2$

Step 2: Set up partial fraction

$$\frac{x^2+x}{(x-1)^2{(x+1)}^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1} + \frac{D}{(x+1)^2}$$

Step 3: Multiply both sides by $(x-1)^2{(x+1)}^2$

$$x^2 + x = A(x-1)(x+1)^2 + B(x+1)^2 + C(x-1)^2(x+1) + D(x-1)^2$$

Step 4: Solve for A, B, C, D

Put $x = 1$:

- $1 + 1 = 2$
- RHS: $B(2)^2 = 4B$
- $4B = 2 \to B = \frac{1}{2}$

Put $x = -1$:

- $1 - 1 = 0$
- RHS: $D(-2)^2 = 4D$
- $4D = 0 \to D = 0$

Step 5: Expand and compare coefficients

$$x^2 + x = A(x-1)(x^2+2x+1) + \frac{1}{2}(x^2+2x+1) + C(x^2-2x+1)(x+1) + 0$$

- $A(x^3 + x^2 - x - 1) + \frac{1}{2}x^2 + x + \frac{1}{2} + C(x^3 - x^2 - x + 1)$

Collect coefficients:

- x^3: $A + C = 0 \to C = -A$
- x^2: $A + \frac{1}{2} - C = 1$
- x^1: $-A + 1 - C = 1$
- Constant: $-A + \frac{1}{2} + C = 0$

From x^1: $-A + 1 + A = 1$
From constant: $-A + \frac{1}{2} - A = 0 \to -2A + \frac{1}{2} = 0 \to A = \frac{1}{4}$
Then $C = -\frac{1}{4}$

Answer: $\frac{x^2+x}{{(x^2-1)}^2} = \frac{1}{4(x-1)} + \frac{1}{2(x-1)^2} - \frac{1}{4(x+1)}$

Question 13: $\mathbf{\frac{3x^2+4x-5}{{(x-1)}^3}}$

Step 1: Set up partial fraction (repeated linear factor)

$$\frac{3x^2+4x-5}{(x-1)^3} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3}$$

Step 2: Multiply both sides by $(x-1)^3$

$$3x^2+4x-5=A(x-1)^2+B(x-1)+C$$

Step 3: Expand RHS

- $A(x^2-2x+1)+Bx-B+C$
- $Ax^2+(-2A+B)x+(A-B+C)$

Step 4: Compare coefficients

- $A=3$
- $-2A+B=4 \to -6+B=4 \to B=10$
- $A-B+C=-5 \to 3-10+C=-5 \to C=2$

Answer: $\frac{3x^2+4x-5}{(x-1)^3}=\frac{3}{x-1}+\frac{10}{(x-1)^2}+\frac{2}{(x-1)^3}$

Question 14: $\frac{\mathbf{1}}{\mathbf{x(x+1)^3}}$

Step 1: Set up partial fraction

$$\frac{1}{x(x+1)^3}=\frac{A}{x}+\frac{B}{x+1}+\frac{C}{(x+1)^2}+\frac{D}{(x+1)^3}$$

Step 2: Multiply both sides by $x(x+1)^3$

$$1=A(x+1)^3+Bx(x+1)^2+Cx(x+1)+Dx$$

Step 3: Solve for A, B, C, D

Put $x=0$:

- $1=A(1)^3=A \to A=1$

Put $x=-1$:

- $1=D(-1) \to D=-1$

Step 4: Expand and compare coefficients

$$1=A(x^3+3x^2+3x+1)+B(x^3+2x^2+x)+C(x^2+x)+Dx1=x^3+3x^2+3x+1+Bx^3+2Bx^2+Bx+Cx^2+Cx-x$$

Collect coefficients:

- x^3: $1+B=0 \to B=-1$
- x^2: $3+2B+C=0 \to 3-2+C=0 \to C=-1$
- x^1: $3+B+C-1=0 \to 3-1-1-1=0$

Answer: $\frac{1}{x(x+1)^3}=\frac{1}{x}-\frac{1}{x+1}-\frac{1}{(x+1)^2}-\frac{1}{(x+1)^3}$

Question 15: $\frac{\mathbf{4x^2-3x+1}}{\mathbf{()^2}}$

Step 1: Set up partial fraction

$$\frac{4x^2-3x+1}{()^2}=\frac{A}{x+1}+\frac{B}{x-1}+\frac{C}{(x-1)^2}$$

Step 2: Multiply both sides by $()^2$

$$4x^2 - 3x + 1 = A(x-1)^2 + B(x+1)(x-1) + C(x+1)$$

Step 3: Solve for A, B, C

Put $x = 1$:

- $4 - 3 + 1 = 2$
- RHS: $C(2) = 2C$
- $2C = 2 \rightarrow C = 1$

Put $x = -1$:

- $4 + 3 + 1 = 8$
- RHS: $A(-2)^2 = 4A$
- $4A = 8 \rightarrow A = 2$

Step 4: Expand and compare coefficients

$$4x^2 - 3x + 1 = 2(x^2 - 2x + 1) + B(x^2 - 1) + 1(x+1) = 2x^2 - 4x + 2 + Bx^2 - B + x + 1 = (2+B)x^2 + (-3)x + (3-B)$$

Compare x^2: $4 = 2 + B \rightarrow B = 2$

Answer: $\frac{4x^2-3x+1}{()^2} = \frac{2}{x+1} + \frac{2}{x-1} + \frac{1}{(x-1)^2}$

Question 16: $\mathbf{\frac{12x^2-48}{(x-2)^2(x+2)^2}}$

Step 1: Factor numerator

- $12x^2 - 48 = 12(x^2 - 4) = 12(x-2)(x+2)$

Step 2: Set up partial fraction

$$\frac{12(x-2)(x+2)}{(x-2)^2(x+2)^2} = \frac{12}{(x-2)(x+2)}$$

Step 3: Decompose

$$\frac{12}{(x-2)(x+2)} = \frac{A}{x-2} + \frac{B}{x+2}$$

Step 4: Multiply both sides by $(x-2)(x+2)$

$$12 = A(x+2) + B(x-2)$$

Put $x = 2$:

- $12 = 4A \rightarrow A = 3$

Put $x = -2$:

- $12 = -4B \rightarrow B = -3$

Answer: $\frac{12x^2-48}{(x-2)^2(x+2)^2} = \frac{3}{x-2} - \frac{3}{x+2}$

Summary Table

1	$\frac{2}{a-b}\left(\frac{1}{x-a} - \frac{1}{x-b}\right)$
2	$1 - \frac{1}{x+1} + \frac{1}{x-1}$
3	$\frac{2x+3}{x^2+4x+5}$
4	$\frac{5}{x+1} - \frac{40}{x+2} + \frac{44}{x+3}$
5	$\frac{2}{x+1} - \frac{4}{x+2} + \frac{2}{x+3}$
6	$\frac{5}{2}\left(\frac{1}{x-1} - \frac{1}{x+1}\right)$
7	$\frac{1}{x-1} + \frac{1}{x-2} + \frac{1}{x-3}$
8	$\frac{9}{2(x-1)} - \frac{9}{x-2} + \frac{9}{2(x-3)}$
9	$\frac{3}{x-4} - \frac{6}{x-5} + \frac{3}{x-6}$
10	General form given
11	$\frac{1}{x-1} + \frac{2}{(x-1)^2}$
12	$\frac{1}{4(x-1)} + \frac{1}{2(x-1)^2} - \frac{1}{4(x+1)}$
13	$\frac{3}{x-1} + \frac{10}{(x-1)^2} + \frac{2}{(x-1)^3}$
14	$\frac{1}{x} - \frac{1}{x+1} - \frac{1}{(x+1)^2} - \frac{1}{(x+1)^3}$
15	$\frac{2}{x+1} + \frac{2}{x-1} + \frac{1}{(x-1)^2}$
16	$\frac{3}{x-2} - \frac{3}{x+2}$