EXERCISE 5.2 - PARTIAL FRACTIONS

Complete Solutions

Question 1: $\mathbf{\frac{2x^2+3x+3}{(x+1)(x^2+1)}}$

Step 1: Denominator has linear factor $(x + 1)$ and irreducible quadratic $(x^2 + 1)$

Step 2: Set up partial fraction decomposition

$$\frac{2x^2 + 3x + 3}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx + C}{x^2 + 1}$$

Step 3: Multiply both sides by $(x + 1)(x^2 + 1)$

$$2x^2 + 3x + 3 = A(x^2 + 1) + (Bx + C)(x + 1)$$

Step 4: Expand RHS

$$2x^2 + 3x + 3 = Ax^2 + A + Bx^2 + Bx + Cx + C = (A + B)x^2 + (B + C)x + (A + C)$$

Step 5: Compare coefficients

- x^2: $A + B = 2$ ... (1)
- x: $B + C = 3$ ... (2)
- Constant: $A + C = 3$ ... (3)

Step 6: Solve the system
From (1): $B = 2 - A$
From (3): $C = 3 - A$
Substitute in (2): $(2 - A) + (3 - A) = 3$

$$5 - 2A = 3 – 2A = −2A = 1$$

Then:

$$B = 2 - 1 = 1 C = 3 - 1 = 2$$

Answer: $\frac{2x^2+3x+3}{(x+1)(x^2+1)} = \frac{1}{x+1} + \frac{x+2}{x^2+1}$

Question 2: $\mathbf{\frac{2x+1}{(x-2)(x^2+3x+5)}}$

Step 1: Check discriminant of quadratic

- $x^2 + 3x + 5$: discriminant $= 9 - 20 = -11 < 0$ (irreducible)

Step 2: Set up partial fraction

$$\frac{2x + 1}{(x-2)(x^2+3x+5)} = \frac{A}{x-2} + \frac{Bx + C}{x^2 + 3x + 5}$$

Step 3: Multiply both sides by $(x - 2)(x^2 + 3x + 5)$

$$2x + 1 = A(x^2 + 3x + 5) + (Bx + C)(x - 2)$$

Step 4: Expand RHS

$$2x+1 = Ax^2+3Ax+5A+Bx^2-2Bx+Cx-2C = (A+B)x^2+(3A-2B+C)x+(5A-2C)$$

Step 5: Compare coefficients

- x^2: $A+B=0$ ... (1)
- x: $3A-2B+C=2$ ... (2)
- Constant: $5A-2C=1$ ... (3)

Step 6: Solve the system
From (1): $B=-A$
From (3): $5A-2C=1 \to 2C=5A-1 \to C=\frac{5A-1}{2}$

Substitute in (2): $3A-2(-A)+\frac{5A-1}{2}=2$

$$3A+2A+\frac{5A-1}{2}=25A+\frac{5A-1}{2}=2\frac{10A+5A-1}{2}=215A-1=415A=5A=\frac{1}{3}$$

Then:

$$B=-\frac{1}{3}C=\frac{5(\frac{1}{3})-1}{2}=\frac{\frac{5}{3}-1}{2}=\frac{\frac{2}{3}}{2}=\frac{1}{3}$$

Answer: $\frac{2x+1}{(x-2)(x^2+3x+5)}=\frac{1}{3(x-2)}+\frac{-\frac{1}{3}x+\frac{1}{3}}{x^2+3x+5}=\frac{1}{3(x-2)}-\frac{x-1}{3(x^2+3x+5)}$

Question 3: $\mathbf{\frac{2x+32}{(x^2+2)^2}}$

Step 1: Denominator has linear factor $(x-2)$ and repeated irreducible quadratic $(x^2+2)^2$

Step 2: Set up partial fraction

$$\frac{2x+32}{(x^2+2)^2}=\frac{A}{x-2}+\frac{Bx+C}{x^2+2}+\frac{Dx+E}{(x^2+2)^2}$$

Step 3: Multiply both sides by $(x^2+2)^2$

$$2x+32=A(x^2+2)^2+(Bx+C)(x-2)(x^2+2)+(Dx+E)(x-2)$$

Step 4: Expand RHS

$$2x+32=A(x^4+4x^2+4)+(Bx+C)(x^3-2x^2+2x-4)+(Dx+E)(x-2)$$

Step 5: Expand further

$$=A(x^4+4x^2+4)+B(x^4-2x^3+2x^2-4x)+C(x^3-2x^2+2x-4)+D(x^2-2x)+E(x-2)$$

Step 6: Collect coefficients

- x^4: $A+B=0$ ... (1)
- x^3: $-2B+C=0$ ... (2)
- x^2: $4A+2B-2C+D=0$ ... (3)
- x: $-4B+2C-2D+E=2$ ... (4)

- Constant: $4A - 4C - 2E = 32$ ... (5)

Step 7: Solve the system
From (1): $B = -A$
From (2): $C = 2B = -2A$

Substitute in (3): $4A + 2(-A) - 2(-2A) + D = 0$

$$4A - 2A + 4A + D = 06A + D = 0$$

$\rightarrow D = -6A$

Substitute in (4): $-4(-A) + 2(-2A) - 2(-6A) + E = 2$

$$4A - 4A + 12A + E = 212A + E = 2$$

$\rightarrow E = 2 - 12A$

Substitute in (5): $4A - 4(-2A) - 2(2 - 12A) = 32$

$$4A + 8A - 4 + 24A = 3236A - 4 = 3236A = 36A = 1$$

Then:

$$B = -1C = -2D = -6E = 2 - 12 = -10$$

Answer: $\frac{2x+32}{(x^2+2)^2} = \frac{1}{x-2} + \frac{-x-2}{x^2+2} + \frac{-6x-10}{(x^2+2)^2}$

$$= \frac{1}{x-2} - \frac{x+2}{x^2+2} - \frac{6x+10}{(x^2+2)^2}$$

Question 4: $\mathbf{\frac{3x^2+3}{x^3+1}}$

Step 1: Factor denominator

- $x^3 + 1 = (x+1)(x^2 - x + 1)$

Step 2: Set up partial fraction

$$\frac{3x^2+3}{(x+1)(x^2-x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}$$

Step 3: Multiply both sides by $(x+1)(x^2-x+1)$

$$3x^2 + 3 = A(x^2 - x + 1) + (Bx + C)(x + 1)$$

Step 4: Expand RHS

$$3x^2 + 3 = Ax^2 - Ax + A + Bx^2 + Bx + Cx + C = (A+B)x^2 + (-A+B+C)x + (A+C)$$

Step 5: Compare coefficients

- x^2: $A + B = 3$ ... (1)
- x: $-A + B + C = 0$ ... (2)
- Constant: $A + C = 3$ ... (3)

Step 6: Solve the system
From (3): $C = 3 - A$
From (1): $B = 3 - A$

Substitute in (2): $-A + (3 - A) + (3 - A) = 0$

$$-A + 3 - A + 3 - A = 06 - 3A = 0A = 2$$

Then:

$$B = 3 - 2 = 1C = 3 - 2 = 1$$

Answer: $\frac{3x^2+3}{x^3+1} = \frac{2}{x+1} + \frac{x+1}{x^2-x+1}$

Question 5: $\mathbf{\frac{x^4}{x^4+2x^2+1}}$

Step 1: Note that degree of numerator = degree of denominator

- Need to divide first

Step 2: Factor denominator

- $x^4 + 2x^2 + 1 = (x^2 + 1)^2$

Step 3: Perform division

$$\frac{x^4}{x^4 + 2x^2 + 1} = 1 - \frac{2x^2 + 1}{x^4 + 2x^2 + 1} = 1 - \frac{2x^2 + 1}{(x^2 + 1)^2}$$

Step 4: Decompose $\frac{2x^2+1}{(x^2+1)^2}$

$$\frac{2x^2 + 1}{(x^2 + 1)^2} = \frac{Ax + B}{x^2 + 1} + \frac{Cx + D}{(x^2 + 1)^2}$$

Step 5: Multiply both sides by $(x^2 + 1)^2$

$$2x^2 + 1 = (Ax + B)(x^2 + 1) + (Cx + D)= Ax^3 + Bx^2 + Ax + B + Cx + D= Ax^3 + Bx^2 + (A + C)x + (B + D)$$

Step 6: Compare coefficients

- x^3: $A = 0$
- x^2: $B = 2$
- x: $A + C = 0 \to 0 + C = 0 \to C = 0$
- Constant: $B + D = 1 \to 2 + D = 1 \to D = -1$

Step 7: Substitute back

$$\frac{2x^2 + 1}{(x^2 + 1)^2} = \frac{2}{x^2 + 1} - \frac{1}{(x^2 + 1)^2}$$

Step 8: Final result

$$\frac{x^4}{x^4+2x^2+1} = 1 - \frac{2}{x^2+1} + \frac{1}{\left(x^2+1\right)^2}$$

Answer: $\frac{x^4}{\left(x^2+1\right)^2} = 1 - \frac{2}{x^2+1} + \frac{1}{\left(x^2+1\right)^2}$

Question 6: $\mathbf{\frac{6x^4+40x^2}{(x^2)\left(x^2+4\right)^2}}$

Step 1: Note: $4 - x^2 = -(x^2-4) = -(x-2)(x+2)$

Step 2: Factor numerator

- $6x^4 + 40x^2 = 2x^2(3x^2+20)$

Step 3: Check degree

- Numerator degree: 4
- Denominator degree: $2+4=6$
- Proper fraction

Step 4: Write in terms of x^2

$$\frac{6x^4+40x^2}{(x^2)\left(x^2+4\right)^2} = \frac{6x^4+40x^2}{-(x^2-4)(x^2+4)^2}$$

Step 5: Since denominator has $(x^2-4)\left(x^2+4\right)^2$, we can use substitution $t = x^2$

Step 6: Set up partial fraction in terms of $t = x^2$

$$\frac{6t^2+40t}{()^2}$$

Step 7: Decompose

$$\frac{6t^2+40t}{()^2} = \frac{A}{4-t} + \frac{B}{t+4} + \frac{C}{(t+4)^2}$$

Step 8: Note: $4 - t = -(t-4)$, so $\frac{A}{4-t} = -\frac{A}{t-4}$

Step 9: Multiply both sides by $()^2$

$$6t^2 + 40t = A(t+4)^2 + B(4-t)(t+4) + C(4-t)$$

Step 10: Expand RHS

$$= A(t^2+8t+16) + B(4t+16-t^2-4t) + C(4-t) = A(t^2+8t+16) + B(16-t^2) + C(4-t) = (A-B)t^2 + (8A-C)t + (1$$

Step 11: Compare coefficients

- t^2: $A - B = 6$ ... (1)
- t: $8A - C = 40$ ... (2)
- Constant: $16A + 16B + 4C = 0$ ... (3)

Step 12: Solve the system
From (1): $B = A - 6$
From (2): $C = 8A - 40$

Substitute in (3): $16A + 16(A - 6) + 4(8A - 40) = 0$

$$16A + 16A - 96 + 32A - 160 = 064A - 256 = 0A = 4$$

Then:

$$B = 4 - 6 = -2C = 32 - 40 = -8$$

Step 13: Substitute back

$$\frac{6t^2 + 40t}{()^2} = \frac{4}{4-t} - \frac{2}{t+4} - \frac{8}{(t+4)^2}$$

Step 14: Replace $t = x^2$

$$\frac{6x^4 + 40x^2}{(x^2)\left(x^2+4\right)^2} = \frac{4}{4-x^2} - \frac{2}{x^2+4} - \frac{8}{\left(x^2+4\right)^2}$$

Step 15: Decompose $\frac{4}{4-x^2}$ further

$$\frac{4}{4-x^2} = \frac{4}{(2-x)(2+x)} = \frac{A}{2-x} + \frac{B}{2+x} 4 = A(2+x) + B(2-x)$$

Put $x = 2$: $4 = 4A \to A = 1$
Put $x = -2$: $4 = 4B \to B = 1$

So $\frac{4}{4-x^2} = \frac{1}{2-x} + \frac{1}{2+x}$

Answer: $\frac{6x^4+40x^2}{(x^2)\left(x^2+4\right)^2} = \frac{1}{2-x} + \frac{1}{2+x} - \frac{2}{x^2+4} - \frac{8}{\left(x^2+4\right)^2}$

Summary Table

1	$\frac{1}{x+1} + \frac{x+2}{x^2+1}$
2	$\frac{1}{3(x-2)} - \frac{x-1}{3(x^2+3x+5)}$
3	$\frac{1}{x-2} - \frac{x+2}{x^2+2} - \frac{6x+10}{\left(x^2+2\right)^2}$
4	$\frac{2}{x+1} + \frac{x+1}{x^2-x+1}$
5	$1 - \frac{2}{x^2+1} + \frac{1}{\left(x^2+1\right)^2}$
6	$\frac{1}{2-x} + \frac{1}{2+x} - \frac{2}{x^2+4} - \frac{8}{\left(x^2+4\right)^2}$