EXERCISE 6.10 - SUMS OF SERIES

Complete Solutions

Question 1: Sum the following series up to n terms

(i) $\mathbf{1 \times 3 + 2 \times 5 + 3 \times 7 + \ldots}$

Step 1: Identify the general term

- First factor: 1, 2, 3, ... $\rightarrow k$
- Second factor: 3, 5, 7, ... $\rightarrow 2k+1$
- $T_k = k(2k+1) = 2k^2 + k$

Step 2: Sum to n terms

$$S_n = \sum_{k=1}^{n} (2k^2 + k) = 2\sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} k$$

Step 3: Use formulas

- $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$
- $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

$$S_n = 2 \cdot \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} = \frac{n(n+1)(2n+1)}{3} + \frac{n(n+1)}{2} = n(n+1)\left(\frac{2n+1}{3} + \frac{1}{2}\right) = n(n+1)\left(\frac{4n+2+3}{6}\right) =$$

Answer: $S_n = \frac{n(n+1)(4n+5)}{6}$

(ii) $\mathbf{1 \times 5 + 2 \times 8 + 3 \times 11 + \ldots}$

Step 1: General term

- First factor: k
- Second factor: 5, 8, 11, ... $\rightarrow 3k+2$
- $T_k = k(3k+2) = 3k^2 + 2k$

Step 2: Sum

$$S_n = \sum_{k=1}^{n} (3k^2 + 2k) = 3\sum_{k=1}^{n} k^2 + 2\sum_{k=1}^{n} k = 3 \cdot \frac{n(n+1)(2n+1)}{6} + 2 \cdot \frac{n(n+1)}{2} = \frac{n(n+1)(2n+1)}{2} + n(n+1) = n(n+1)\left(\right.$$

Answer: $S_n = \frac{n(n+1)(2n+3)}{2}$

(iii) $\mathbf{1 \times 2 + 2 \times 5 + 3 \times 8 + \ldots}$

Step 1: General term

- First factor: k
- Second factor: 2, 5, 8, ... $\rightarrow 3k-1$
- $T_k = k(3k-1) = 3k^2 - k$

Step 2: Sum

$$S_n = \sum_{k=1}^{n}(3k^2 - k) = 3\sum_{k=1}^{n}k^2 - \sum_{k=1}^{n}k = 3 \cdot \frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} = \frac{n(n+1)(2n+1)}{2} - \frac{n(n+1)}{2} = \frac{n(n+1)}{2}(2n+1-$$

Answer: $S_n = n^2(n+1)$

(iv) $\mathbf{1 \times 3 \times 5 + 2 \times 4 \times 6 + 3 \times 5 \times 7 + \ldots}$

Step 1: General term

- First factor: k
- Second factor: $k+2$
- Third factor: $k+4$
- $T_k = k(k+2)(k+4)$

Step 2: Expand

$$T_k = k(k^2 + 6k + 8) = k^3 + 6k^2 + 8k$$

Step 3: Sum

$$S_n = \sum_{k=1}^{n}(k^3 + 6k^2 + 8k) = \sum_{k=1}^{n}k^3 + 6\sum_{k=1}^{n}k^2 + 8\sum_{k=1}^{n}k$$

Step 4: Use formulas

- $\sum k^3 = \left[\frac{n(n+1)}{2}\right]^2$

$$S_n = \frac{n^2(n+1)^2}{4} + 6 \cdot \frac{n(n+1)(2n+1)}{6} + 8 \cdot \frac{n(n+1)}{2} = \frac{n^2(n+1)^2}{4} + n(n+1)(2n+1) + 4n(n+1) = n(n+1)\left[\frac{n(n+1)}{4} + \right.$$

Answer: $S_n = \frac{n(n+1)(n^2+9n+20)}{4}$

(v) $\mathbf{1 \times 2 \times 4 + 2 \times 3 \times 7 + 3 \times 4 \times 10 + \ldots}$

Step 1: General term

- First factor: k
- Second factor: $k+1$
- Third factor: $4, 7, 10, \ldots \to 3k+1$
- $T_k = k(k+1)(3k+1)$

Step 2: Expand

$$T_k = (k^2 + k)(3k+1) = 3k^3 + k^2 + 3k^2 + k = 3k^3 + 4k^2 + k$$

Step 3: Sum

$$S_n = 3\sum k^3 + 4\sum k^2 + \sum k = 3\cdot\frac{n^2(n+1)^2}{4} + 4\cdot\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} = \frac{3n^2(n+1)^2}{4} + \frac{2n(n+1)(2n+1)}{3} + \frac{n(n}{}$$

Answer: $S_n = \frac{n(n+1)(9n^2+25n+14)}{12}$

(vi) $\mathbf{2^2+4^2+6^2+\ldots}$

Step 1: General term: $(2k)^2 = 4k^2$

Step 2: Sum

$$S_n = \sum_{k=1}^{n} 4k^2 = 4\cdot\frac{n(n+1)(2n+1)}{6} = \frac{2n(n+1)(2n+1)}{3}$$

Answer: $S_n = \frac{2n(n+1)(2n+1)}{3}$

(vii) $\mathbf{3^2+6^2+9^2+\ldots}$

Step 1: General term: $(3k)^2 = 9k^2$

Step 2: Sum

$$S_n = \sum_{k=1}^{n} 9k^2 = 9\cdot\frac{n(n+1)(2n+1)}{6} = \frac{3n(n+1)(2n+1)}{2}$$

Answer: $S_n = \frac{3n(n+1)(2n+1)}{2}$

(viii) $\mathbf{4\times1^2+7\times2^2+10\times3^2+\ldots}$

Step 1: General term

- First factor: 4, 7, 10, ... $\to 3k+1$
- Second factor: k^2
- $T_k = (3k+1)k^2 = 3k^3 + k^2$

Step 2: Sum

$$S_n = 3\sum k^3 + \sum k^2 = 3\cdot\frac{n^2(n+1)^2}{4} + \frac{n(n+1)(2n+1)}{6} = \frac{n(n+1)}{2}\left[\frac{3n(n+1)}{2} + \frac{2n+1}{3}\right] = \frac{n(n+1)}{6}[9n(n+1)+2(2n+$$

Answer: $S_n = \frac{n(n+1)(9n^2+13n+2)}{6}$

(ix) $\mathbf{3+(3+7)+(3+7+11)+\ldots}$

Step 1: The terms are sums of A.P. 3, 7, 11, ... (common difference 4)

- kth term: sum of first k terms of 3, 7, 11, ...
- First term of inner A.P. $= 3$, $d = 4$
- $T_k = \frac{k}{2}[2(3)+(k-1)4] = \frac{k}{2}(6+4k-4) = \frac{k}{2}(4k+2) = k(2k+1) = 2k^2+k$

Step 2: Sum

$$S_n = \sum_{k=1}^{n} (2k^2 + k) = 2\sum k^2 + \sum k = 2 \cdot \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} = \frac{n(n+1)(2n+1)}{3} + \frac{n(n+1)}{2} = \frac{n(n+1)(4n+5)}{6}$$

Answer: $S_n = \frac{n(n+1)(4n+5)}{6}$

(x) $\mathbf{1^2+(1^2+2^2)+(1^2+2^2+3^2)+\ldots}$

Step 1: kth term = sum of squares from 1 to k

$$T_k = \frac{k(k+1)(2k+1)}{6} = \frac{2k^3+3k^2+k}{6}$$

Step 2: Sum

$$S_n = \sum_{k=1}^{n} \frac{2k^3+3k^2+k}{6} = \frac{1}{6}\left[2\sum k^3 + 3\sum k^2 + \sum k\right] = \frac{1}{6}\left[2 \cdot \frac{n^2(n+1)^2}{4} + 3 \cdot \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}\right] = \frac{1}{6}\left[\frac{n^2(n+}{2}\right.$$

Answer: $S_n = \frac{n(n+1)^2(n+2)}{12}$

Question 2: Sum the series

(i) $\mathbf{1^2-2^2+3^2-4^2+\cdots+(2n-1)^2-(2n)^2}$

Step 1: Group terms in pairs

$$S = (1^2-2^2)+(3^2-4^2)+\cdots+((2n-1)^2-(2n)^2)$$

Step 2: Each pair: $(2k-1)^2-(2k)^2 = (4k^2-4k+1)-4k^2 = -4k+1$

Step 3: Sum

$$S = \sum_{k=1}^{n}(-4k+1) = -4\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1 = -4 \cdot \frac{n(n+1)}{2} + n = -2n(n+1)+n = -2n^2-2n+n = -2n^2-n$$

Answer: $S = -n(2n+1)$

(ii) $\mathbf{1^2+1^2+2^2+1^2+2^2+3^2+\ldots}$

Step 1: The series is: $1^2+(1^2+2^2)+(1^2+2^2+3^2)+\ldots$

- This is exactly the same as Question 1(x)

Answer: $S_n = \frac{n(n+1)^2(n+2)}{12}$

(iii) $\mathbf{1+2+3+\cdots+n}$

Answer: $S_n = \frac{n(n+1)}{2}$

(iv) $\mathbf{1+2+3+\cdots+n}$

Answer: $S_n = \frac{n(n+1)}{2}$

Question 3: Find the sum to n terms whose nth term is given

(i) $\mathbf{a_n = 5n^2 + 2n + 3}$

$$S_n = \sum_{k=1}^{n}(5k^2 + 2k + 3) = 5\sum k^2 + 2\sum k + 3\sum 1 = 5 \cdot \frac{n(n+1)(2n+1)}{6} + 2 \cdot \frac{n(n+1)}{2} + 3n = \frac{5n(n+1)(2n+1)}{6} + n(n +$$

Answer: $S_n = \frac{n(10n^2+21n+29)}{6}$

(ii) $\mathbf{a_n = n^2 + 2n - 3}$

$$S_n = \sum_{k=1}^{n}(k^2 + 2k - 3) = \sum k^2 + 2\sum k - 3\sum 1 = \frac{n(n+1)(2n+1)}{6} + 2 \cdot \frac{n(n+1)}{2} - 3n = \frac{n(n+1)(2n+1)}{6} + n(n+1) - 3$$

Answer: $S_n = \frac{n(2n^2+9n-11)}{6}$

Question 4: Given nth terms, find sum to 2n terms

(i) $\mathbf{a_n = 3n^2 + 5n + 2}$

Step 1: Sum to m terms

$$S_m = \sum_{k=1}^{m}(3k^2 + 5k + 2) = 3\sum k^2 + 5\sum k + 2\sum 1 = 3 \cdot \frac{m(m+1)(2m+1)}{6} + 5 \cdot \frac{m(m+1)}{2} + 2m = \frac{m(m+1)(2m+1)}{2} +$$

Step 2: Put $m = 2n$

$$S_{2n} = 2n(4n^2 + 8n + 5)$$

Answer: $S_{2n} = 2n(4n^2 + 8n + 5)$

(ii) $\mathbf{a_n = n^2 + n - 2}$

Step 1: Sum to m terms

$$S_m = \sum_{k=1}^{m}(k^2 + k - 2) = \sum k^2 + \sum k - 2\sum 1 = \frac{m(m+1)(2m+1)}{6} + \frac{m(m+1)}{2} - 2m = \frac{m(m+1)(2m+1) + 3m(m+1) -}{6}$$

Step 2: Put $m = 2n$

$$S_{2n} = \frac{2n(4n^2 + 6n - 4)}{3} = \frac{2n(2n^2 + 3n - 2)}{3}$$

Answer: $S_{2n} = \frac{2n(2n^2+3n-2)}{3}$

Summary Table

1(i)	$\frac{n(n+1)(4n+5)}{6}$
1(ii)	$\frac{n(n+1)(2n+3)}{2}$
1(iii)	$n^2(n+1)$
1(iv)	$\frac{n(n+1)(n^2+9n+20)}{4}$
1(v)	$\frac{n(n+1)(9n^2+25n+14)}{12}$
1(vi)	$\frac{2n(n+1)(2n+1)}{3}$
1(vii)	$\frac{3n(n+1)(2n+1)}{2}$
1(viii)	$\frac{n(n+1)(9n^2+13n+2)}{6}$
1(ix)	$\frac{n(n+1)(4n+5)}{6}$
1(x)	$\frac{n(n+1)^2(n+2)}{12}$
2(i)	$-n(2n+1)$
2(ii)	$\frac{n(n+1)^2(n+2)}{12}$
2(iii)	$\frac{n(n+1)}{2}$
2(iv)	$\frac{n(n+1)}{2}$
3(i)	$\frac{n(10n^2+21n+29)}{6}$
3(ii)	$\frac{n(2n^2+9n-11)}{6}$
4(i)	$2n(4n^2+8n+5)$
4(ii)	$\frac{2n(2n^2+3n-2)}{3}$