EXERCISE 6.4 - ARITHMETIC SERIES

Complete Solutions

Question 1: Sum the series

(i) $\mathbf{3+6+9+\cdots+a_{20}}$

Step 1: Identify the sequence

- $a_1 = 3, d = 3$

Step 2: Find a_{20}

- $a_{20} = 3 + (20-1)(3) = 3 + 57 = 60$

Step 3: Sum formula $S_n = \frac{n}{2}(a_1 + a_n)$

- $S_{20} = \frac{20}{2}(3 + 60) = 10(63) = 630$

Answer: 630

(ii) $\mathbf{\frac{4}{\sqrt{5}}+\frac{\sqrt{5}}{5}+\frac{6}{\sqrt{5}}+\cdots+a_n}$

Step 1: Identify the pattern

- $a_1 = \frac{4}{\sqrt{5}}$
- $a_2 = \frac{\sqrt{5}}{5} = \frac{1}{\sqrt{5}}$
- $a_3 = \frac{6}{\sqrt{5}}$

Step 2: Find common difference

- $d = a_2 - a_1 = \frac{1}{\sqrt{5}} - \frac{4}{\sqrt{5}} = -\frac{3}{\sqrt{5}}$
- Check: $a_3 = a_2 + d = \frac{1}{\sqrt{5}} - \frac{3}{\sqrt{5}} = -\frac{2}{\sqrt{5}}$ — This doesn't match $\frac{6}{\sqrt{5}}$

Re-evaluate: The sequence might be:

- $\frac{4}{\sqrt{5}}, \frac{5}{\sqrt{5}}, \frac{6}{\sqrt{5}}, \ldots$
- $\frac{4}{\sqrt{5}}, \sqrt{5}, \frac{6}{\sqrt{5}}, \ldots$

Step 3: $a_1 = \frac{4}{\sqrt{5}}, d = \frac{1}{\sqrt{5}}$

- $a_n = \frac{4}{\sqrt{5}} + (n-1)\frac{1}{\sqrt{5}} = \frac{n+3}{\sqrt{5}}$

Step 4: Sum formula

$$S_n = \frac{n}{2}\left(\frac{4}{\sqrt{5}} + \frac{n+3}{\sqrt{5}}\right) = \frac{n}{2}\left(\frac{n+7}{\sqrt{5}}\right) = \frac{n(n+7)}{2\sqrt{5}}$$

Answer: $S_n = \frac{n(n+7)}{2\sqrt{5}}$

Question 2: Find $\mathbf{S_n}$ for each arithmetic series

(i) $\mathbf{a_1 = 4, n = 25, a_n = 100}$

Step 1: Use $S_n = \frac{n}{2}(a_1 + a_n)$

- $S_{25} = \frac{25}{2}(4 + 100) = \frac{25}{2}(104) = 25 \times 52 = 1300$

Answer: 1300

(ii) $a_1 = 40, n = 20, d = -3$

Step 1: Find a_n

- $a_{20} = 40 + 19(-3) = 40 - 57 = -17$

Step 2: Sum

- $S_{20} = \frac{20}{2}(40 + (-17)) = 10(23) = 230$

Answer: 230

(iii) $a_n = 52, n = 21, d = -4$

Step 1: Find a_1

- $a_n = a_1 + (n-1)d$
- $52 = a_1 + 20(-4) = a_1 - 80$
- $a_1 = 132$

Step 2: Sum

- $S_{21} = \frac{21}{2}(132 + 52) = \frac{21}{2}(184) = 21 \times 92 = 1932$

Answer: 1932

Question 3: Find a_1 for $d = 8, n = 19, S_n = 1786$

Step 1: Use $S_n = \frac{n}{2}[2a_1 + (n-1)d]$

- $1786 = \frac{19}{2}[2a_1 + 18(8)]$
- $1786 = \frac{19}{2}(2a_1 + 144)$
- $3572 = 19(2a_1 + 144)$
- $3572 = 38a_1 + 2736$
- $38a_1 = 836$
- $a_1 = 22$

Answer: $a_1 = 22$

Question 4: How many terms of $96 + 93 + 90 + \ldots$ amount to 1071?

Step 1: Identify sequence

- $a_1 = 96, d = -3$

Step 2: Use $S_n = \frac{n}{2}[2a_1 + (n-1)d] = 1071$

- $\frac{n}{2}[192 + (n-1)(-3)] = 1071$
- $\frac{n}{2}[192 - 3n + 3] = 1071$
- $\frac{n}{2}(195 - 3n) = 1071$
- $n(195 - 3n) = 2142$
- $195n - 3n^2 = 2142$
- $3n^2 - 195n + 2142 = 0$
- $n^2 - 65n + 714 = 0$

Step 3: Factor

- $(n-21)(n-34)=0$
- $n=21$ or $n=34$

Step 4: Check both

- For $n=21$: $S_{21}=\frac{21}{2}(96+96+20(-3))=\frac{21}{2}(192-60)=\frac{21}{2}(132)=1386\neq 1071$
- Actually, let's recalculate: $S_{21}=21(96+21\times(-3))=21(96-63)=21(33)=693$

Wait, let's check properly:

For $n=21$: $S_{21}=\frac{21}{2}[2(96)+20(-3)]=\frac{21}{2}(192-60)=\frac{21}{2}(132)=1386$
For $n=34$: $S_{34}=\frac{34}{2}[192+33(-3)]=17(192-99)=17(93)=1581$

Neither gives 1071! Let's redo the quadratic:

Step 3 (corrected):

$$n(195-3n)=2142195n-3n^2=21423n^2-195n+2142=0$$

Divide by 3: $n^2-65n+714=0$

Check: $21\times 34=714$, $21+34=55\neq 65$

So $n^2-65n+714=(n-14)(n-51)=0$
$n=14$ or $n=51$

Check: $14+51=65$, $14\times 51=714$

For $n=14$: $S_{14}=\frac{14}{2}[192+13(-3)]=7(192-39)=7(153)=1071$

Answer: 14 terms

Question 5: Sides of right triangle in A.P., perimeter 36 cm

Step 1: Let sides be $a-d, a, a+d$ (in A.P.)

- Hypotenuse is $a+d$

Step 2: Perimeter: $(a-d)+a+(a+d)=3a=36$

- $a=12$

Step 3: Pythagorean theorem

$$(a-d)^2+a^2=(a+d)^2(12-d)^2+144=(12+d)^2144-24d+d^2+144=144+24d+d^2288-24d=144+24d144=48d$$

Step 4: Sides: $12-3=9, 12, 12+3=15$

Answer: 9 cm, 12 cm, 15 cm

Question 6: Sum series in groups of 3

(i) $3+5-7+9+11-13+15+17-19+\ldots$ to 3n terms

Step 1: Group in sets of 3:

- Group 1: $3+5-7=1$
- Group 2: $9+11-13=7$

- Group 3: $15 + 17 - 19 = 13$

Step 2: The group sums form an A.P. with $d = 6$

Step 3: Sum of 3n terms = sum of n group sums

- Group sums: $1, 7, 13, \ldots$
- $S_n = \frac{n}{2}[2(1) + (n-1)6] = \frac{n}{2}(6n - 4) = n(3n - 2)$

Answer: $3n^2 - 2n$

(ii) $1 + 4 - 7 + 10 + 13 - 16 + 19 + 22 - 25 + \ldots$ to 3n terms

Step 1: Group in sets of 3:

- Group 1: $1 + 4 - 7 = -2$
- Group 2: $10 + 13 - 16 = 7$
- Group 3: $19 + 22 - 25 = 16$

Step 2: Group sums: $-2, 7, 16, \ldots$ (A.P. with $d = 9$)

Step 3: Sum $= \frac{n}{2}[2(-2) + (n-1)9] = \frac{n}{2}(9n - 13)$

Answer: $\frac{9n^2 - 13n}{2}$

Question 7: Sum of 20 terms, $\mathbf{r^{th}}$ term $= 3r + 1$

Step 1: $a_r = 3r + 1$

- $a_1 = 4, a_2 = 7, a_3 = 10, \ldots$
- This is an A.P. with $d = 3$

Step 2: Sum of 20 terms

- $S_{20} = \frac{20}{2}[2(4) + 19(3)] = 10(8 + 57) = 10(65) = 650$

Answer: 650

Question 8: 5th term = 11, 9th term = 17

Step 1: Set up equations

- $a_5 = a_1 + 4d = 11 \ldots (1)$
- $a_9 = a_1 + 8d = 17 \ldots (2)$

Step 2: Subtract (1) from (2)

- $4d = 6 \to d = \frac{3}{2}$

Step 3: $a_1 = 11 - 4(\frac{3}{2}) = 11 - 6 = 5$

Step 4: Sum of 20 terms

- $S_{20} = \frac{20}{2}[2(5) + 19(\frac{3}{2})] = 10[10 + \frac{57}{2}] = 10(\frac{77}{2}) = 385$

Answer: 385

Question 9: Sum of integers in first 1000 not divisible by 2 or 5

Step 1: Numbers not divisible by 2: odd numbers: $1, 3, 5, \ldots, 999$

Step 2: From these, remove numbers divisible by 5: $5, 15, 25, \ldots, 995$

Step 3: Sum of odd numbers from 1 to 999:

- Number of terms: 500
- $S_{\text{odd}} = \frac{500}{2}(1+999) = 250(1000) = 250000$

Step 4: Odd numbers divisible by 5: $5, 15, 25, \dots, 995$

- These are numbers: $5(1), 5(3), 5(5), \dots, 5(199)$
- Number of terms: 100
- Sum: $5(1+3+5+\cdots+199) = 5 \times 100^2 = 50000$
  (Sum of first 100 odd numbers $= 100^2 = 10000$)

Step 5: Required sum

- $250000 - 50000 = 200000$

Answer: 200,000

Question 10: Sum of 9 terms = 171, 8th term = 31

Step 1: $S_9 = 171$

$$\frac{9}{2}[2a_1 + 8d] = 1719(2a_1 + 8d) = 342$$

$2a_1 + 8d = 38$ ... (1)

Step 2: $a_8 = a_1 + 7d = 31$ ... (2)

Step 3: From (1): $a_1 + 4d = 19$
Subtract from (2): $3d = 12 \to d = 4$

Step 4: $a_1 = 31 - 7(4) = 31 - 28 = 3$

Answer: 3, 7, 11, 15, ...

Question 11: 5th term = 21, sum of first 6 terms = 90

Step 1: $a_5 = a_1 + 4d = 21$ ... (1)
Step 2: $S_6 = \frac{6}{2}[2a_1 + 5d] = 90$

$$3(2a_1 + 5d) = 90$$

$2a_1 + 5d = 30$ ... (2)

Step 3: Multiply (1) by 2: $2a_1 + 8d = 42$
Subtract (2): $3d = 12 \to d = 4$

Step 4: $a_1 = 21 - 16 = 5$

Step 5: $a_{18} = 5 + 17(4) = 5 + 68 = 73$

Answer: 73

Question 12: Three numbers in A.P., sum = 24, product = 440

Step 1: Let numbers be $a-d, a, a+d$
- Sum: $3a = 24 \to a = 8$

Step 2: Product: $(8-d)(8)(8+d) = 440$
- $8(64 - d^2) = 440$
- $64 - d^2 = 55$

- $d^2 = 9$
- $d = \pm 3$

Step 3: Numbers: $5, 8, 11$ or $11, 8, 5$

Answer: 5, 8, 11 (or 11, 8, 5)

Question 13: A.P. 2, 6, 10, 14, ... find least n for sum > 2000

Step 1: $a_1 = 2, d = 4$
- $S_n = \frac{n}{2}[2(2) + (n-1)4] = \frac{n}{2}(4n) = 2n^2$

Step 2: Need $2n^2 > 2000 \rightarrow n^2 > 1000$
- $n > \sqrt{1000} \approx 31.62$

Step 3: Least integer $n = 32$

Check: $S_{31} = 2(31)^2 = 1922$

$$S_{32} = 2(32)^2 = 2048 > 2000$$

Answer: 32 terms

Question 14: Four numbers in A.P., sum = 32, sum of squares = 276

Step 1: Let numbers be $a - 3d, a - d, a + d, a + 3d$

Step 2: Sum: $4a = 32 \rightarrow a = 8$

Step 3: Sum of squares:

$$(8-3d)^2 + (8-d)^2 + (8+d)^2 + (8+3d)^2 = 276 \;\; (64 - 48d + 9d^2) + (64 - 16d + d^2) + (64 + 16d + d^2) + (64 + 48d + 9d^2) =$$

Step 4: Numbers: $5, 7, 9, 11$ (or $11, 9, 7, 5$)

Answer: 5, 7, 9, 11

Question 15: Five numbers in A.P., sum = 25, sum of squares = 135

Step 1: Let numbers be $a - 2d, a - d, a, a + d, a + 2d$

Step 2: Sum: $5a = 25 \rightarrow a = 5$

Step 3: Sum of squares:

$$(5-2d)^2 + (5-d)^2 + 5^2 + (5+d)^2 + (5+2d)^2 = 135 \;\; (25 - 20d + 4d^2) + (25 - 10d + d^2) + 25 + (25 + 10d + d^2) + (25 + 20$$

Step 4: Numbers: $3, 4, 5, 6, 7$ (or $7, 6, 5, 4, 3$)

Answer: 3, 4, 5, 6, 7

Question 16: If $\frac{1}{a+b} + \frac{1}{c+a} + \frac{1}{b+c} = \frac{1}{a+b+c}$ are in A.P.

Note: The problem statement seems to have a typo. It should be:
"If $\frac{1}{a+b}, \frac{1}{c+a}, \frac{1}{b+c}$ are in A.P. then show that a^2, b^2, c^2 are in A.P."

Step 1: In A.P., $2 \times \text{middle} = \text{first} + \text{last}$

$$\frac{2}{c+a} = \frac{1}{a+b} + \frac{1}{b+c}$$

Step 2: Simplify RHS

$$\frac{2}{a+c} = \frac{(b+c)+(a+b)}{(a+b)(b+c)} = \frac{a+c+2b}{(a+b)(b+c)}$$

Step 3: Cross multiply

$$2(a+b)(b+c) = (a+c)(a+c+2b)2(ab+ac+b^2+bc) = (a+c)^2 + 2b(a+c)2ab+2ac+2b^2+2bc = a^2+2ac+c^2+2ab$$

Step 4: Cancel common terms

- $2ab$ cancels, $2ac$ cancels, $2bc$ cancels

$$2b^2 = a^2 + c^2$$

Step 5: $2b^2 = a^2 + c^2$ means a^2, b^2, c^2 are in A.P.

Therefore, a^2, b^2, c^2 are in A.P.

Summary Table

1(i)	630
1(ii)	$\frac{n(n+7)}{2\sqrt{5}}$
2(i)	1300
2(ii)	230
2(iii)	1932
3	$a_1 = 22$
4	14 terms
5	9 cm, 12 cm, 15 cm
6(i)	$3n^2 - 2n$
6(ii)	$\frac{9n^2-13n}{2}$
7	650
8	385
9	200,000
10	3, 7, 11, 15, ...
11	73
12	5, 8, 11
13	32 terms
14	5, 7, 9, 11
15	3, 4, 5, 6, 7
16	Proven

EXERCISE 6.4 - ARITHMETIC SERIES (Continued)

Question 17: Sum of first four terms = 56, sum of last four terms = 112, first term = 11

Step 1: Let the A.P. have n terms with first term $a_1 = 11$ and common difference d.

Step 2: Sum of first 4 terms:

$$S_4 = \frac{4}{2}[2a_1 + 3d] = 2[22 + 3d] = 5644 + 6d = 566d = 12d = 2$$

Step 3: The last four terms are: $a_{n-3}, a_{n-2}, a_{n-1}, a_n$

- Their sum $= \frac{4}{2}[a_{n-3} + a_n] = 2[a_{n-3} + a_n] = 112$
- So $a_{n-3} + a_n = 56$

Step 4: Express in terms of a_1, d, n:

- $a_n = a_1 + (n-1)d = 11 + 2(n-1) = 2n + 9$
- $a_{n-3} = a_1 + (n-4)d = 11 + 2(n-4) = 2n + 3$

Step 5: Sum:

$$(2n+3) + (2n+9) = 564n + 12 = 564n = 44n = 11$$

Answer: 11 terms

Question 18: First, second and last terms are a, b and c

Step 1: In an A.P., first term $a_1 = a$, second term $a_2 = b$

- Common difference $d = b - a$

Step 2: Last term $a_n = c$

- $c = a + (n-1)(b-a)$
- $(n-1)(b-a) = c - a$
- $n - 1 = \frac{c-a}{b-a}$
- $n = \frac{c-a}{b-a} + 1 = \frac{c-a+b-a}{b-a} = \frac{b+c-2a}{b-a}$

Step 3: Sum of A.P.

$$S_n = \frac{n}{2}(a+c) = \frac{1}{2} \cdot \frac{b+c-2a}{b-a} \cdot (a+c) = \frac{(b+c-2a)(a+c)}{2(b-a)}$$

Answer: $S = \frac{(b+c-2a)(c+a)}{2(b-a)}$

Question 19: Sum of n A.Ms. between a and b is n times the single A.M.

Step 1: Single A.M. between a and b is:

$$A = \frac{a+b}{2}$$

Step 2: If n arithmetic means are inserted between a and b, the sequence is:

$$a, A_1, A_2, \ldots, A_n, b$$

- Total terms: $n + 2$
- Common difference $d = \frac{b-a}{n+1}$

Step 3: The arithmetic means are:

$$A_k = a + kd = a + \frac{k(b-a)}{n+1}, k = 1, 2, \ldots, n$$

Step 4: Sum of the n means:

$$S = \sum_{k=1}^{n}\left(a + \frac{k(b-a)}{n+1}\right) = na + \frac{b-a}{n+1}\sum_{k=1}^{n} k = na + \frac{b-a}{n+1} \cdot \frac{n(n+1)}{2} = na + \frac{n(b-a)}{2} = \frac{2na+nb-na}{2} = \frac{na+nb}{2} = \frac{n(a+b)}{2}$$

Step 5: Compare with n times single A.M.:

$$n \cdot A = n \cdot \frac{a+b}{2} = \frac{n(a+b)}{2}$$

Therefore, sum of n A.Ms. = $n\times$ (single A.M.)

Summary Table

17	11 terms
18	$\frac{(b+c-2a)(c+a)}{2(b-a)}$
19	Proven