EXERCISE 6.6 - GEOMETRIC MEANS

Complete Solutions

Question 1: Find G.M. between the given numbers

(i) -2 and 8

Step 1: Geometric Mean formula

$$G = \pm\sqrt{ab}$$

Step 2: Substitute

$$G = \pm\sqrt{(-2)(8)} = \pm\sqrt{-16} = \pm 4i$$

Note: Since the product is negative, the G.M. is imaginary.

Answer: $\pm 4i$

(ii) -2i and 8i

Step 1: Substitute

$$G = \pm\sqrt{(-2i)(8i)} = \pm\sqrt{-16i^2} = \pm\sqrt{-16(-1)} = \pm\sqrt{16} = \pm 4$$

Answer: ± 4

(iii) 6 and 9

Step 1: Substitute

$$G = \pm\sqrt{6 \times 9} = \pm\sqrt{54} = \pm 3\sqrt{6}$$

Answer: $\pm 3\sqrt{6}$

Question 2: Insert four real geometric means between 3 and 96

Step 1: We need sequence: $3, G_1, G_2, G_3, G_4, 96$

- Total terms: 6
- First term $a_1 = 3$
- Last term $a_6 = 96$

Step 2: For G.P., $a_n = a_1 r^{n-1}$

- $96 = 3r^{6-1} = 3r^5$
- $r^5 = 32$
- $r = 2$

Step 3: Find the geometric means

- $G_1 = 3 \times 2 = 6$
- $G_2 = 6 \times 2 = 12$
- $G_3 = 12 \times 2 = 24$

- $G_4 = 24 \times 2 = 48$

Check: $48 \times 2 = 96$

Answer: 6, 12, 24, 48

Question 3: G.M. between x and y is less than their A.M.

Step 1: Given $x > 0, y > 0, x \neq y$

- G.M. $= \sqrt{xy}$
- A.M. $= \frac{x+y}{2}$

Step 2: We need to prove $\sqrt{xy} < \frac{x+y}{2}$

Step 3: Square both sides (both are positive)

$$xy < \frac{(x+y)^2}{4} 4xy < x^2 + 2xy + y^2 0 < x^2 - 2xy + y^2 0 < (x-y)^2$$

Step 4: Since $x \neq y$, we have $(x-y)^2 > 0$, which is true.

Therefore, G.M. < A.M. for positive distinct real numbers

Question 4: For what value of n, $\frac{a^n + b^n}{a^{n-1} + b^{n-1}}$ is the positive G.M. between a and b?

Step 1: Positive G.M. between a and b is $\sqrt{ab}$

Step 2: Set up equation

$$\frac{a^n + b^n}{a^{n-1} + b^{n-1}} = \sqrt{ab}$$

Step 3: Cross multiply

$$a^n + b^n = \sqrt{ab}(a^{n-1} + b^{n-1}) a^n + b^n = a^{n-\frac{1}{2}} b^{\frac{1}{2}} + a^{\frac{1}{2}} b^{n-\frac{1}{2}}$$

Step 4: Rearrange

$$a^n - a^{n-\frac{1}{2}} b^{\frac{1}{2}} + b^n - a^{\frac{1}{2}} b^{n-\frac{1}{2}} = 0 a^{n-\frac{1}{2}}(a^{\frac{1}{2}} - b^{\frac{1}{2}}) + b^{n-\frac{1}{2}}(b^{\frac{1}{2}} - a^{\frac{1}{2}}) = 0 (a^{\frac{1}{2}} - b^{\frac{1}{2}})(a^{n-\frac{1}{2}} - b^{n-\frac{1}{2}}) = 0$$

Step 5: Since $a \neq b$, we have $a^{\frac{1}{2}} - b^{\frac{1}{2}} \neq 0$

- So $a^{n-\frac{1}{2}} - b^{n-\frac{1}{2}} = 0$
- $a^{n-\frac{1}{2}} = b^{n-\frac{1}{2}}$

Step 6: For $a \neq b$, this holds when $n - \frac{1}{2} = 0$

- $n = \frac{1}{2}$

Check: For $n = \frac{1}{2}$:

$\frac{a^{1/2} + b^{1/2}}{a^{-1/2} + b^{-1/2}} = \frac{\sqrt{a} + \sqrt{b}}{\frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}}} = \frac{\sqrt{a} + \sqrt{b}}{\frac{\sqrt{a} + \sqrt{b}}{\sqrt{ab}}} = \sqrt{ab}$

Answer: $n = \frac{1}{2}$

Summary Table

1(i)	$\pm 4i$
1(ii)	± 4
1(iii)	$\pm 3\sqrt{6}$
2	6, 12, 24, 48
3	Proven: G.M. $<$ A.M.
4	$n = \frac{1}{2}$

EXERCISE 6.6 - GEOMETRIC MEANS (Continued)

Question 5: A.M. exceeds G.M. by 2, sum is 20

Step 1: Let the numbers be x and y (positive integers)

- A.M. $= \frac{x+y}{2} = \frac{20}{2} = 10$
- G.M. $= \sqrt{xy}$

Step 2: A.M. - G.M. = 2

$$10 - \sqrt{xy} = 2\sqrt{xy} = 8xy = 64$$

Step 3: We have:

- $x + y = 20$
- $xy = 64$

Step 4: Solve the system

- $y = 20 - x$
- $x(20 - x) = 64$
- $20x - x^2 = 64$
- $x^2 - 20x + 64 = 0$
- $(x - 4)(x - 16) = 0$
- $x = 4$ or $x = 16$

Step 5: If $x = 4, y = 16$; if $x = 16, y = 4$

Answer: 4 and 16

Question 6: A.M. = 5, G.M. = 4

Step 1: Let numbers be x and y

- A.M. $= \frac{x+y}{2} = 5 \rightarrow x + y = 10$
- G.M. $= \sqrt{xy} = 4 \rightarrow xy = 16$

Step 2: Solve: $y = 10 - x$

- $x(10 - x) = 16$
- $10x - x^2 = 16$
- $x^2 - 10x + 16 = 0$
- $(x - 2)(x - 8) = 0$

- $x = 2$ or $x = 8$

Answer: 2 and 8

Question 7: A.M. is double G.M.

Step 1: Let numbers be a and b

- A.M. $= \frac{a+b}{2}$
- G.M. $= \sqrt{ab}$

Step 2: Given: A.M. $= 2 \times$ G.M.

$$\frac{a+b}{2} = 2\sqrt{ab}a + b = 4\sqrt{ab}$$

Step 3: Let $r = \frac{a}{b}$

- $a = rb$
- $rb + b = 4\sqrt{rb \cdot b} = 4b\sqrt{r}$
- $b(r+1) = 4b\sqrt{r}$
- $r + 1 = 4\sqrt{r}$

Step 4: Let $t = \sqrt{r}$

- $t^2 + 1 = 4t$
- $t^2 - 4t + 1 = 0$
- $t = \frac{4\pm\sqrt{16-4}}{2} = \frac{4\pm\sqrt{12}}{2} = \frac{4\pm2\sqrt{3}}{2} = 2 \pm \sqrt{3}$

Step 5: Since $r = t^2$

$$\frac{a}{b} = (2 \pm \sqrt{3})^2 = 7 \pm 4\sqrt{3}$$

Step 6: The stated ratio is $a : b = 2 + \sqrt{3} : 2 - \sqrt{3}$

Check: $\frac{2+\sqrt{3}}{2-\sqrt{3}} \cdot \frac{2+\sqrt{3}}{2+\sqrt{3}} = \frac{(2+\sqrt{3})^2}{4-3} = 7 + 4\sqrt{3}$

Therefore, $a : b = 2 + \sqrt{3} : 2 - \sqrt{3}$

Question 8: One G.M. and two A.Ms. inserted between two positive numbers

Step 1: Let the two positive numbers be x and y

- One G.M. between them: $G = \sqrt{xy}$
- Two A.Ms. between them: p and q

Step 2: Sequence with two A.Ms.: x, p, q, y

- In A.P.: $p - x = q - p = y - q = d$
- $p = x + d$, $q = x + 2d$, $y = x + 3d$

Step 3: Express x and y in terms of p and q:

- From $p = x + d$ and $q = x + 2d$:
- $d = q - p$

- $x = p - d = p - (q - p) = 2p - q$
- $y = q + d = q + (q - p) = 2q - p$

Step 4: Since G is the G.M. between x and y:

$$G^2 = xy = (2p - q)(2q - p)$$

Therefore, $G^2 = (2p - q)(2q - p)$

Summary Table

5	4 and 16
6	2 and 8
7	Proven: $a : b = 2 + \sqrt{3} : 2 - \sqrt{3}$
8	Proven: $G^2 = (2p - q)(2q - p)$