EXERCISE 6.7 - SUM OF GEOMETRIC SERIES

Complete Solutions

Question 1: Sum of first 15 terms of $\mathbf{1, \frac{1}{3}, \frac{1}{9}, \ldots}$

Step 1: Identify first term and common ratio

- $a_1 = 1$
- $r = \frac{1}{3}$

Step 2: Use sum formula for G.P. ($r \neq 1$)

$$S_n = \frac{a_1(1 - r^n)}{1 - r}$$

Step 3: For $n = 15$

$$S_{15} = \frac{1\left(1 - \left(\frac{1}{3}\right)^{15}\right)}{1 - \frac{1}{3}} = \frac{1 - \frac{1}{3^{15}}}{\frac{2}{3}} = \frac{3}{2}\left(1 - \frac{1}{3^{15}}\right)$$

Answer: $S_{15} = \frac{3}{2}\left(1 - \frac{1}{3^{15}}\right)$

Question 2: 3rd term = 16, 6th term = -128

Step 1: Use $a_n = a_1 r^{n-1}$

- $a_3 = a_1 r^2 = 16 \ldots$ (1)
- $a_6 = a_1 r^5 = -128 \ldots$ (2)

Step 2: Divide (2) by (1)

$$\frac{a_1 r^5}{a_1 r^2} = \frac{-128}{16} r^3 = -8 r = -2$$

Step 3: Find a_1

- From (1): $a_1(-2)^2 = 16 \rightarrow 4a_1 = 16 \rightarrow a_1 = 4$

Step 4: Sum of first 7 terms

$$S_7 = \frac{a_1(1 - r^7)}{1 - r} = \frac{4(1 - (-2)^7)}{1 - (-2)} = \frac{4(1 - (-128))}{3} = \frac{4(129)}{3} = \frac{516}{3} = 172$$

Answer: $a_1 = 4$, $S_7 = 172$

Question 3: Sum to n terms

(i) $\mathbf{0.2 + 0.22 + 0.222 + \ldots}$

Step 1: Write as:

$$0.2 + 0.22 + 0.222 + \cdots = \frac{2}{10} + \frac{22}{100} + \frac{222}{1000} + \cdots = 2\left(\frac{1}{10} + \frac{11}{100} + \frac{111}{1000} + \cdots\right)$$

Step 2: Alternative approach:

$$0.2 + 0.22 + 0.222 + \cdots = 2\left[\frac{1}{10} + \frac{11}{100} + \frac{111}{1000} + \cdots\right]$$

Step 3: Write each term as:

- $0.2 = \frac{2}{9}(1 - \frac{1}{10})$
- $0.22 = \frac{2}{9}(1 - \frac{1}{10^2})$
- $0.222 = \frac{2}{9}(1 - \frac{1}{10^3})$

Step 4: Sum:

$$S_n = \frac{2}{9}\left[\left(1 - \frac{1}{10}\right) + \left(1 - \frac{1}{10^2}\right) + \cdots + \left(1 - \frac{1}{10^n}\right)\right] = \frac{2}{9}\left[n - \left(\frac{1}{10} + \frac{1}{10^2} + \cdots + \frac{1}{10^n}\right)\right] = \frac{2}{9}\left[n - \frac{\frac{1}{10}\left(1 - \frac{1}{10^n}\right)}{1 - \frac{1}{10}}\right] = \frac{2}{9}\left[n - \frac{1}{9}\left(\right.\right.$$

Answer: $S_n = \frac{2}{9}\left[n - \frac{1}{9}\left(1 - \frac{1}{10^n}\right)\right]$

(ii) $\mathbf{3 + 33 + 333 + \ldots}$

Step 1: Write as:

$$3 + 33 + 333 + \cdots = 3(1 + 11 + 111 + \cdots) = 3\left[\frac{10 - 1}{9} + \frac{10^2 - 1}{9} + \frac{10^3 - 1}{9} + \cdots\right]$$

Step 2: Simplify

$$= \frac{3}{9}\left[(10 - 1) + (10^2 - 1) + (10^3 - 1) + \cdots + (10^n - 1)\right] = \frac{1}{3}\left[(10 + 10^2 + 10^3 + \cdots + 10^n) - n\right]$$

Step 3: Sum of geometric series $10 + 10^2 + \cdots + 10^n$

$$= \frac{10(10^n - 1)}{10 - 1} = \frac{10(10^n - 1)}{9}$$

Step 4: Final sum

$$S_n = \frac{1}{3}\left[\frac{10(10^n - 1)}{9} - n\right]$$

Answer: $S_n = \frac{10(10^n - 1)}{27} - \frac{n}{3}$

Question 4: Sum to n terms

(i) $\mathbf{1 + (a + b) + (a^2 + ab + b^2) + (a^3 + a^2 b + ab^2 + b^3) + \ldots}$

Step 1: Observe the pattern:

- Term 1: $1 = \frac{a^1 - b^1}{a - b}$
- Term 2: $a + b = \frac{a^2 - b^2}{a - b}$
- Term 3: $a^2 + ab + b^2 = \frac{a^3 - b^3}{a - b}$
- Term k: $a^{k-1} + a^{k-2}b + \cdots + b^{k-1} = \frac{a^k - b^k}{a - b}$

Step 2: Sum of first n terms

$$S_n = \frac{1}{a-b}\left[(a-b)+(a^2-b^2)+(a^3-b^3)+\cdots+(a^n-b^n)\right] = \frac{1}{a-b}\left[(a+a^2+\cdots+a^n)-(b+b^2+\cdots+b^n)\right]$$

Step 3: Sum the geometric series

$$S_n = \frac{1}{a-b}\left[\frac{a(a^n-1)}{a-1}-\frac{b(b^n-1)}{b-1}\right]$$

Answer: $S_n = \frac{1}{a-b}\left[\frac{a(a^n-1)}{a-1}-\frac{b(b^n-1)}{b-1}\right]$

(ii) $\mathbf{r+(1+k)r^2+(1+k+k^2)r^3+\ldots}$

Step 1: Term m: $(1+k+k^2+\cdots+k^{m-1})r^m$

Step 2: Write as:

$$S_n = \sum_{m=1}^{n}\left(\frac{k^m-1}{k-1}\right)r^m = \frac{1}{k-1}\sum_{m=1}^{n}\left[(kr)^m-r^m\right]$$

Step 3: Sum the geometric series

$$S_n = \frac{1}{k-1}\left[\frac{kr((kr)^n-1)}{kr-1}-\frac{r(r^n-1)}{r-1}\right]$$

Answer: $S_n = \frac{1}{k-1}\left[\frac{kr((kr)^n-1)}{kr-1}-\frac{r(r^n-1)}{r-1}\right]$

Question 5: Sum $\mathbf{2+(1-i)+\left(\frac{1}{i}\right)+\ldots}$ to 8 terms

Step 1: Identify first term and common ratio

- $a_1 = 2$
- $a_2 = 1-i$
- $r = \frac{a_2}{a_1} = \frac{1-i}{2}$

Step 2: Check: $a_3 = a_2\cdot r = (1-i)\frac{1-i}{2} = \frac{(1-i)^2}{2} = \frac{1-2i+i^2}{2} = \frac{1-2i-1}{2} = -i = \frac{1}{i}$

Step 3: Sum of 8 terms

$$S_8 = \frac{a_1(1-r^8)}{1-r} = \frac{2\left(1-\left(\frac{1-i}{2}\right)^8\right)}{1-\frac{1-i}{2}}$$

Step 4: Simplify denominator

$$1-\frac{1-i}{2} = \frac{2-1+i}{2} = \frac{1+i}{2}$$

Step 5: Numerator factor

$$2\left(1-\frac{(1-i)^8}{2^8}\right) = 2-\frac{(1-i)^8}{128}$$

Step 6: $(1-i)^2 = -2i$

- $(1-i)^4 = (-2i)^2 = -4$
- $(1-i)^8 = (-4)^2 = 16$

Step 7: Numerator

$$2 - \frac{16}{128} = 2 - \frac{1}{8} = \frac{15}{8}$$

Step 8: Final sum

$$S_8 = \frac{\frac{15}{8}}{\frac{1+i}{2}} = \frac{15}{8} \cdot \frac{2}{1+i} = \frac{15}{4(1+i)} = \frac{15(1-i)}{4(1+i)(1-i)} = \frac{15(1-i)}{4(2)} = \frac{15(1-i)}{8}$$

Answer: $S_8 = \frac{15}{8}(1-i)$

Question 6: Ratio of sum of first n terms to sum from $(n+1)$th to $(2n)$th

Step 1: Let G.P. have first term a and common ratio r

- Sum of first n terms: $S_1 = \frac{a(1-r^n)}{1-r}$

Step 2: Sum from term $(n+1)$ to $(2n)$:

$$S_2 = a_{n+1} + a_{n+2} + \cdots + a_{2n} = ar^n + ar^{n+1} + \cdots + ar^{2n-1} = ar^n(1 + r + r^2 + \cdots + r^{n-1}) = ar^n \cdot \frac{1-r^n}{1-r}$$

Step 3: Ratio

$$\frac{S_1}{S_2} = \frac{\frac{a(1-r^n)}{1-r}}{\frac{ar^n(1-r^n)}{1-r}} = \frac{1}{r^n}$$

Answer: The ratio is $\frac{1}{r^n}$

Summary Table

1	$\frac{3}{2}\left(1 - \frac{1}{3^{15}}\right)$
2	$a_1 = 4, S_7 = 172$
3(i)	$\frac{2}{9}\left[n - \frac{1}{9}\left(1 - \frac{1}{10^n}\right)\right]$
3(ii)	$\frac{10(10^n-1)}{27} - \frac{n}{3}$
4(i)	$\frac{1}{a-b}\left[\frac{a(a^n-1)}{a-1} - \frac{b(b^n-1)}{b-1}\right]$
4(ii)	$\frac{1}{k-1}\left[\frac{kr((kr)^n-1)}{kr-1} - \frac{r(r^n-1)}{r-1}\right]$
5	$\frac{15}{8}(1-i)$
6	$\frac{1}{r^n}$