EXERCISE 6.8 - ARITHMETIC-GEOMETRIC SEQUENCES

Complete Solutions

Question 1: Find the 8th term of the arithmetic-geometric sequence

Step 1: Arithmetic part: 1, 4, 7, ...

- $a_n = 1 + (n-1)3 = 3n - 2$

Step 2: Geometric part: 5, 10, 20, ...

- $b_n = 5 \cdot 2^{n-1}$

Step 3: The arithmetic-geometric sequence terms are $a_n \cdot b_n$

- $c_n = (3n-2) \cdot 5 \cdot 2^{n-1}$

Step 4: Find 8th term

- $c_8 = (3(8)-2) \cdot 5 \cdot 2^{8-1} = (24-2) \cdot 5 \cdot 2^7 = 22 \cdot 5 \cdot 128 = 22 \cdot 640 = 14080$

Answer: 14080

Question 2: Find the nth term

Step 1: Arithmetic part: 3, 7, 11, ...

- $a_n = 3 + (n-1)4 = 4n - 1$

Step 2: Geometric part: 2, 6, 18, ...

- $b_n = 2 \cdot 3^{n-1}$

Step 3: nth term of arithmetic-geometric sequence

- $c_n = a_n \cdot b_n = (4n-1) \cdot 2 \cdot 3^{n-1}$

Answer: $c_n = 2(4n-1)3^{n-1}$

Question 3: Arithmetic part: $\mathbf{a_{n+1} = 2n + 5}$, Geometric part: $\mathbf{b_{n+2} = \frac{1}{9}(-3)^n}$

Step 1: Find arithmetic part as a function of n

- $a_{n+1} = 2n + 5$
- Let $k = n + 1$, then $n = k - 1$
- $a_k = 2(k-1) + 5 = 2k - 2 + 5 = 2k + 3$
- So $a_n = 2n + 3$

Step 2: Find geometric part as a function of n

- $b_{n+2} = \frac{1}{9}(-3)^n$
- Let $k = n + 2$, then $n = k - 2$
- $b_k = \frac{1}{9}(-3)^{k-2} = \frac{1}{9} \cdot \frac{(-3)^k}{(-3)^2} = \frac{1}{9} \cdot \frac{(-3)^k}{9} = \frac{(-3)^k}{81}$
- So $b_n = \frac{(-3)^n}{81}$

Step 3: nth term of arithmetic-geometric sequence

- $c_n = a_n \cdot b_n = (2n+3) \cdot \frac{(-3)^n}{81}$

Step 4: Sum of first three terms

- $c_1 = (2+3) \cdot \frac{(-3)^1}{81} = 5 \cdot \frac{-3}{81} = -\frac{15}{81} = -\frac{5}{27}$

- $c_2 = (4+3) \cdot \frac{(-3)^2}{81} = 7 \cdot \frac{9}{81} = 7 \cdot \frac{1}{9} = \frac{7}{9}$
- $c_3 = (6+3) \cdot \frac{(-3)^3}{81} = 9 \cdot \frac{-27}{81} = 9 \cdot \left(-\frac{1}{3}\right) = -3$

Step 5: Sum

$$S_3 = -\frac{5}{27} + \frac{7}{9} - 3 = -\frac{5}{27} + \frac{21}{27} - \frac{81}{27} = -\frac{65}{27}$$

Answer: $c_n = \frac{()^n}{81}$, $S_3 = -\frac{65}{27}$

Question 4: Sum to n terms

(i) $\mathbf{1 \cdot 2 + 3 \cdot 4 + 5 \cdot 8 + 7 \cdot 16 + \dots}$

Step 1: Arithmetic part: 1, 3, 5, 7, ... $\to a_n = 2n - 1$
Step 2: Geometric part: 2, 4, 8, 16, ... $\to b_n = 2^n$
Step 3: General term: $c_n = (2n-1)2^n$

Step 4: Sum to n terms

$$S_n = \sum_{k=1}^{n} (2k-1)2^k = 2\sum_{k=1}^{n} k2^k - \sum_{k=1}^{n} 2^k$$

Step 5: Use formulas:

- $\sum_{k=1}^{n} 2^k = 2(2^n - 1)$
- $\sum_{k=1}^{n} k2^k = (n-1)2^{n+1} + 2$

Step 6: Substitute

$S_n = 2[(n-1)2^{n+1} + 2] - 2(2^n - 1) = 2(n-1)2^{n+1} + 4 - 2^{n+1} + 2 = (n-1)2^{n+2} - 2^{n+1} + 6 = 2^{n+1}(2n-2-1) + 6 = 2^{n+1}(2$

Answer: $S_n = 2^{n+1}(2n-3) + 6$

(ii) $\mathbf{2 \cdot 3 + 4{\cdot}3^2 + 6{\cdot}3^3 + 8{\cdot}3^4 + \dots}$

Step 1: Arithmetic part: 2, 4, 6, 8, ... $\to a_n = 2n$
Step 2: Geometric part: 3, 3^2, 3^3, 3^4, ... $\to b_n = 3^n$
Step 3: General term: $c_n = 2n \cdot 3^n$

Step 4: Sum to n terms

$$S_n = \sum_{k=1}^{n} 2k{\cdot}3^k = 2\sum_{k=1}^{n} k3^k$$

Step 5: Use formula: $\sum_{k=1}^{n} kr^k = \frac{r[1-(n+1)r^n + nr^{n+1}]}{(1-r)^2}$

- For $r = 3$: $\sum_{k=1}^{n} k3^k = \frac{3[1-(n+1)3^n + n3^{n+1}]}{(1-3)^2} = \frac{3[1-(n+1)3^n + 3n3^n]}{4}$

Step 6: Simplify

$$S_n = 2 \cdot \frac{3[1-(n+1)3^n + 3n3^n]}{4} = \frac{3[1+(2n-1)3^n]}{2}$$

Answer: $S_n = \frac{3}{2}[1 + (2n - 1)3^n]$

(iii) $\mathbf{2 + \frac{5}{4} + \frac{8}{4^2} + \frac{11}{4^3} + \ldots}$

Step 1: Arithmetic part (numerators): 2, 5, 8, 11, ... $\rightarrow a_n = 3n - 1$
Step 2: Geometric part: $1, \frac{1}{4}, \frac{1}{4^2}, \frac{1}{4^3}, \ldots \rightarrow b_n = \frac{1}{4^{n-1}}$
Step 3: General term: $c_n = \frac{3n-1}{4^{n-1}}$

Step 4: Sum to n terms

$$S_n = \sum_{k=1}^{n} \frac{3k - 1}{4^{k-1}} = 3 \sum_{k=1}^{n} \frac{k}{4^{k-1}} - \sum_{k=1}^{n} \frac{1}{4^{k-1}}$$

Step 5: Use the formula for A.G. series:
For $S = \sum_{k=1}^{n} \frac{k}{r^{k-1}}$:

$$S = \frac{r^2[1 - (n+1)r^{-n} + nr^{-n-1}]}{(r-1)^2}$$

But here $r = 4$, so $\frac{1}{4^{k-1}}$.

Let $S_1 = \sum_{k=1}^{n} \frac{k}{4^{k-1}} = 1 + \frac{2}{4} + \frac{3}{4^2} + \cdots$

Using formula: $S_1 = \frac{16}{9}\left[1 - \frac{n+1}{4^n} + \frac{n}{4^{n+1}}\right]$

And $S_2 = \sum_{k=1}^{n} \frac{1}{4^{k-1}} = \frac{1 - \frac{1}{4^n}}{1 - \frac{1}{4}} = \frac{4}{3}\left(1 - \frac{1}{4^n}\right)$

Step 6: Final sum

$$S_n = 3 \cdot \frac{16}{9}\left[1 - \frac{n+1}{4^n} + \frac{n}{4^{n+1}}\right] - \frac{4}{3}\left(1 - \frac{1}{4^n}\right) = \frac{16}{3}\left[1 - \frac{n+1}{4^n} + \frac{n}{4^{n+1}}\right] - \frac{4}{3} + \frac{4}{3 \cdot 4^n} = 4 - \frac{16}{3}\left(\frac{n+1}{4^n}\right) + \frac{16}{3}\left(\frac{n}{4^{n+1}}\right) + \frac{4}{3 \cdot 4}$$

Answer: $S_n = 4 - \frac{4(n+1)}{4^n}$

(iv) $\mathbf{1 + \frac{3}{5} + \frac{5}{5^2} + \frac{7}{5^3} + \ldots}$

Step 1: Arithmetic part (numerators): 1, 3, 5, 7, ... $\rightarrow a_n = 2n - 1$
Step 2: Geometric part: $1, \frac{1}{5}, \frac{1}{5^2}, \frac{1}{5^3}, \ldots \rightarrow b_n = \frac{1}{5^{n-1}}$
Step 3: General term: $c_n = \frac{2n-1}{5^{n-1}}$

Step 4: Sum to n terms

$$S_n = \sum_{k=1}^{n} \frac{2k - 1}{5^{k-1}} = 2 \sum_{k=1}^{n} \frac{k}{5^{k-1}} - \sum_{k=1}^{n} \frac{1}{5^{k-1}}$$

Using formulas:

$\sum_{k=1}^{n} \frac{k}{r^{k-1}} = \frac{r[1-(n+1)r^{-n}+nr^{-n-1}]}{(r-1)^2}$ with $r = 5$

$$= \frac{5}{16}\left[1 - \frac{n+1}{5^n} + \frac{n}{5^{n+1}}\right] \sum_{k=1}^{n} \frac{1}{5^{k-1}} = \frac{5}{4}\left(1 - \frac{1}{5^n}\right)$$

Step 5: Final sum

$$S_n = 2 \cdot \frac{5}{16}\left[1 - \frac{n+1}{5^n} + \frac{n}{5^{n+1}}\right] - \frac{5}{4}\left(1 - \frac{1}{5^n}\right) = \frac{5}{8}\left[1 - \frac{n+1}{5^n} + \frac{n}{5^{n+1}}\right] - \frac{5}{4} + \frac{5}{4 \cdot 5^n} = \frac{5}{8} - \frac{5}{4} + \frac{1}{5^n}\left[-\frac{5(n+1)}{8} + \frac{5n}{40} + \frac{5}{4}\right] =$$

Answer: $S_n = -\frac{5}{8} + \frac{5-4n}{8 \cdot 5^n}$

(v) $\mathbf{1 + \frac{4}{3} + \frac{7}{9} + \frac{10}{27} + \ldots}$

Step 1: Arithmetic part (numerators): 1, 4, 7, 10, ... $\rightarrow a_n = 3n - 2$
Step 2: Geometric part: $1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots \rightarrow b_n = \frac{1}{3^{n-1}}$
Step 3: General term: $c_n = \frac{3n-2}{3^{n-1}}$

Step 4: Sum to n terms

$$S_n = \sum_{k=1}^{n} \frac{3k-2}{3^{k-1}} = 3\sum_{k=1}^{n} \frac{k}{3^{k-1}} - 2\sum_{k=1}^{n} \frac{1}{3^{k-1}}$$

Using formulas with $r = 3$:

$$\sum_{k=1}^{n} \frac{k}{3^{k-1}} = \frac{3}{4}\left[1 - \frac{n+1}{3^n} + \frac{n}{3^{n+1}}\right] \sum_{k=1}^{n} \frac{1}{3^{k-1}} = \frac{3}{2}\left(1 - \frac{1}{3^n}\right)$$

Step 5: Final sum

$$S_n = 3 \cdot \frac{3}{4}\left[1 - \frac{n+1}{3^n} + \frac{n}{3^{n+1}}\right] - 2 \cdot \frac{3}{2}\left(1 - \frac{1}{3^n}\right) = \frac{9}{4}\left[1 - \frac{n+1}{3^n} + \frac{n}{3^{n+1}}\right] - 3 + \frac{3}{3^n} = \frac{9}{4} - 3 + \frac{1}{3^n}\left[-\frac{9(n+1)}{4} + \frac{9n}{12} + 3\right] = -$$

Answer: $S_n = -\frac{3}{4} + \frac{3-6n}{4 \cdot 3^n}$

Question 5: Sum the following infinite series

(i) $\mathbf{1 + \frac{3}{2} + \frac{5}{4} + \frac{7}{8} + \ldots}$

Step 1: General term: $a_n = \frac{2n-1}{2^{n-1}}$

Step 2: Sum to infinity

$$S_\infty = \sum_{k=1}^{\infty} \frac{2k-1}{2^{k-1}} = 2\sum_{k=1}^{\infty} \frac{k}{2^{k-1}} - \sum_{k=1}^{\infty} \frac{1}{2^{k-1}}$$

Step 3: Use infinite sum formulas:

- $\sum_{k=1}^{\infty} \frac{k}{r^{k-1}} = \frac{r}{(r-1)^2}$ for $\mid r \mid > 1$
- For $r = 2$: $\sum_{k=1}^{\infty} \frac{k}{2^{k-1}} = \frac{2}{(1)^2} = 2$

Step 4:

$$\sum_{k=1}^{\infty} \frac{1}{2^{k-1}} = \frac{1}{1 - \frac{1}{2}} = 2$$

Step 5: Final sum

$$S_\infty = 2(2) - 2 = 4 - 2 = 2$$

Answer: 2

(ii) $\mathbf{2+\frac{5}{3}+\frac{8}{9}+\frac{11}{27}+\ldots}$

Step 1: General term: $a_n = \frac{3n-1}{3^{n-1}}$

Step 2: Sum to infinity

$$S_\infty = \sum_{k=1}^{\infty} \frac{3k-1}{3^{k-1}} = 3\sum_{k=1}^{\infty} \frac{k}{3^{k-1}} - \sum_{k=1}^{\infty} \frac{1}{3^{k-1}}$$

Step 3: For $r = 3$:

- $\sum_{k=1}^{\infty} \frac{k}{3^{k-1}} = \frac{3}{(2)^2} = \frac{3}{4}$
- $\sum_{k=1}^{\infty} \frac{1}{3^{k-1}} = \frac{1}{1-\frac{1}{3}} = \frac{3}{2}$

Step 4: Final sum

$$S_\infty = 3\left(\frac{3}{4}\right) - \frac{3}{2} = \frac{9}{4} - \frac{6}{4} = \frac{3}{4}$$

Answer: $\frac{3}{4}$

Question 6: Show that $\mathbf{2^2 \cdot 4^4 \cdot 8^8 \cdot 16^{16} \ldots \infty = 4}$

Step 1: Write the product

$$P = 2^2 \cdot 4^4 \cdot 8^8 \cdot 16^{16} \cdots$$

Step 2: Express in powers of 2

- $2 = 2^1$, $4 = 2^2$, $8 = 2^3$, $16 = 2^4$, ...
- $P = (2^1)^2 \cdot (2^2)^4 \cdot (2^3)^8 \cdot (2^4)^{16} \cdots$
- $P = 2^{1\cdot 2} \cdot 2^{2\cdot 4} \cdot 2^{3\cdot 8} \cdot 2^{4\cdot 16} \cdots$
- $P = 2^{2+8+24+64+\cdots}$

Step 3: The exponent is $\sum_{n=1}^{\infty} n \cdot 2^n$

Step 4: Sum $\sum_{n=1}^{\infty} n \cdot 2^n$ diverges!

This product diverges to infinity, not 4. There must be a typo.

Let's re-examine: The terms are $2^2, 4^4, 8^8, 16^{16}, \ldots$

- $2^2 = 4$
- $4^4 = (2^2)^4 = 2^8$
- $8^8 = (2^3)^8 = 2^{24}$
- $16^{16} = (2^4)^{16} = 2^{64}$

The product $= 2^{2+8+24+64+\cdots}$ diverges.

The statement as written is incorrect.

Question 7: Show that $\sqrt{4}\cdot\sqrt{16}\cdot\sqrt{64}\cdot\sqrt{256}\ldots\infty = 16$

Step 1: Simplify each term

- $\sqrt{4} = 2$
- $\sqrt{16} = 4$
- $\sqrt{64} = 8$
- $\sqrt{256} = 16$
- So the product is $2 \cdot 4 \cdot 8 \cdot 16 \cdots$

Step 2: Express in powers of 2

$$P = 2^1 \cdot 2^2 \cdot 2^3 \cdot 2^4 \cdots P = 2^{1+2+3+4+\cdots}$$

Step 3: The sum $1 + 2 + 3 + 4 + \cdots$ diverges to infinity.

The product diverges to infinity, not 16.

The statement as written is incorrect.

Summary Table

1	14080
2	$2(4n-1)3^{n-1}$
3	$\frac{()^n}{81}$, $S_3 = -\frac{65}{27}$
4(i)	$2^{n+1}(2n-3)+6$
4(ii)	$\frac{3}{2}[1+(2n-1)3^n]$
4(iii)	$4 - \frac{4(n+1)}{4^n}$
4(iv)	$-\frac{5}{8} + \frac{5-4n}{8\cdot 5^n}$
4(v)	$-\frac{3}{4} + \frac{3-6n}{4\cdot 3^n}$
5(i)	2
5(ii)	$\frac{3}{4}$
6	Statement incorrect (diverges)
7	Statement incorrect (diverges)

EXERCISE 6.8 - ARITHMETIC-GEOMETRIC SERIES (Continued)

Question 8: Sum to n terms: $\mathbf{2 + 4x + 6x^2 + 8x^3 + \ldots}$ where $\mathbf{x \neq 1}$

Step 1: Identify the pattern

- Arithmetic part: 2, 4, 6, 8, ... $\to a_k = 2k$
- Geometric part: 1, x, x^2, x^3, ... $\to b_k = x^{k-1}$
- General term: $c_k = 2k \cdot x^{k-1}$

Step 2: Sum to n terms

$$S_n = \sum_{k=1}^{n} 2kx^{k-1} = 2\sum_{k=1}^{n} kx^{k-1}$$

Step 3: Use the formula for $\sum_{k=1}^{n} kr^{k-1}$

$$\sum_{k=1}^{n} kr^{k-1} = \frac{1-(n+1)r^n + nr^{n+1}}{(1-r)^2}$$

Step 4: Substitute $r = x$

$$S_n = 2 \cdot \frac{1-(n+1)x^n + nx^{n+1}}{(1-x)^2}$$

Answer: $S_n = \frac{2[1-(n+1)x^n+nx^{n+1}]}{(1-x)^2}$

Question 9: Sum to n terms:

$$\frac{2n+1}{2n-1} + 3\left(\frac{2n+1}{2n-1}\right)^2 + 5\left(\frac{2n+1}{2n-1}\right)^3 + \ldots$$

Note: Here n is used both as the number of terms and in the terms themselves. This is ambiguous. Let's re-interpret:

Let $r = \frac{2n+1}{2n-1}$. Then the series is:

$$r + 3r^2 + 5r^3 + 7r^4 + \ldots \text{ to } m \text{ terms}$$

Step 1: General term: $a_k = (2k-1)r^k$, for $k = 1, 2, \ldots, m$

Step 2: Sum to m terms

$$S_m = \sum_{k=1}^{m} (2k-1)r^k = 2\sum_{k=1}^{m} kr^k - \sum_{k=1}^{m} r^k$$

Step 3: Use formulas:

- $\sum_{k=1}^{m} r^k = \frac{r(1-r^m)}{1-r}$
- $\sum_{k=1}^{m} kr^k = \frac{r[1-(m+1)r^m+mr^{m+1}]}{(1-r)^2}$

Step 4: Substitute

$$S_m = 2 \cdot \frac{r[1-(m+1)r^m + mr^{m+1}]}{(1-r)^2} - \frac{r(1-r^m)}{1-r}$$

Step 5: Simplify

$$S_m = \frac{2r[1-(m+1)r^m + mr^{m+1}] - r(1-r^m)(1-r)}{(1-r)^2} = \frac{r[2-2(m+1)r^m + 2mr^{m+1} - (1-r^m)(1-r)]}{(1-r)^2} = \frac{r[2-2(m+1)r^m}{}$$

Answer: $S_m = \frac{r[1+r-(2m+1)r^m+(2m-1)r^{m+1}]}{(1-r)^2}$

Question 10: Prove that

$$1+2\left(1+\frac{1}{n}\right)+3\left(1+\frac{1}{n}\right)^2+\ldots \text{ to } n \text{ terms } = n^2$$

Step 1: Let $r = 1+\frac{1}{n} = \frac{n+1}{n}$

Step 2: The series is:

$$S_n = \sum_{k=1}^{n} kr^{k-1}$$

Step 3: Use the formula for $\sum_{k=1}^{n} kr^{k-1}$

$$S_n = \frac{1-(n+1)r^n + nr^{n+1}}{(1-r)^2}$$

Step 4: Substitute $r = \frac{n+1}{n}$

- $1-r = 1-\frac{n+1}{n} = -\frac{1}{n}$
- $(1-r)^2 = \frac{1}{n^2}$

Step 5: $r^n = \left(\frac{n+1}{n}\right)^n$

$$S_n = \frac{1-(n+1)\left(\frac{n+1}{n}\right)^n + n\left(\frac{n+1}{n}\right)^{n+1}}{\frac{1}{n^2}} = n^2\left[1-(n+1)\left(\frac{n+1}{n}\right)^n + n\left(\frac{n+1}{n}\right)^{n+1}\right]$$

Step 6: Simplify the bracket

$$1-(n+1)\left(\frac{n+1}{n}\right)^n + n\left(\frac{n+1}{n}\right)\left(\frac{n+1}{n}\right)^n = 1-(n+1)\left(\frac{n+1}{n}\right)^n + (n+1)\left(\frac{n+1}{n}\right)^n = 1$$

Step 7: Therefore

$$S_n = n^2$$

Answer: Proven

Question 11: Sum to n terms: $\mathbf{2+5x+8x^2+11x^3+\ldots}$

Step 1: Identify the pattern

- Arithmetic part: 2, 5, 8, 11, ... $\rightarrow a_k = 3k-1$
- Geometric part: 1, x, x^2, x^3, ... $\rightarrow b_k = x^{k-1}$
- General term: $c_k = (3k-1)x^{k-1}$

Step 2: Sum to n terms

$$S_n = \sum_{k=1}^{n}(3k-1)x^{k-1} = 3\sum_{k=1}^{n} kx^{k-1} - \sum_{k=1}^{n} x^{k-1}$$

Step 3: Use formulas:

- $\sum_{k=1}^{n} x^{k-1} = \frac{1-x^n}{1-x}$

- $\sum_{k=1}^{n} kx^{k-1} = \frac{1-(n+1)x^n+nx^{n+1}}{(1-x)^2}$

Step 4: Substitute

$$S_n = 3 \cdot \frac{1-(n+1)x^n+nx^{n+1}}{(1-x)^2} - \frac{1-x^n}{1-x}$$

Step 5: Simplify

$$S_n = \frac{3[1-(n+1)x^n+nx^{n+1}]-(1-x^n)(1-x)}{(1-x)^2} = \frac{3-3(n+1)x^n+3nx^{n+1}-(1-x-x^n+x^{n+1})}{(1-x)^2} = \frac{2+x-3(n+1)x^n+}{(1-}$$

Step 6: Deduce sum to infinity for $|x|<1$

- As $n \to \infty$, $x^n \to 0$ and $x^{n+1} \to 0$
- $S_\infty = \frac{2+x}{(1-x)^2}$

Answer:

$$S_n = \frac{2+x-(3n+2)x^n+(3n-1)x^{n+1}}{(1-x)^2} S_\infty = \frac{2+x}{(1-x)^2} \text{ for } |x|<1$$

Summary Table

8	$\frac{2[1-(n+1)x^n+nx^{n+1}]}{(1-x)^2}$
9	$\frac{r[1+r-(2m+1)r^m+(2m-1)r^{m+1}]}{(1-r)^2}$, where $r = \frac{2n+1}{2n-1}$
10	Proven
11	$S_n = \frac{2+x-(3n+2)x^n+(3n-1)x^{n+1}}{(1-x)^2}$, $S_\infty = \frac{2+x}{(1-x)^2}$