EXERCISE 8.2 - BINOMIAL THEOREM

Complete Solutions

Question 1: Expand using binomial theorem

(i) $\left(\frac{x}{2} - \frac{2}{x^2}\right)^6$

Step 1: Use binomial theorem: $(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r$

- Here $a = \frac{x}{2}$, $b = -\frac{2}{x^2}$, $n = 6$

Step 2: Expand

$$\sum_{r=0}^{6} \binom{6}{r} \left(\frac{x}{2}\right)^{6-r} \left(-\frac{2}{x^2}\right)^r$$

Step 3: Calculate each term:

- $r = 0$: $\binom{6}{0}\left(\frac{x}{2}\right)^6 = \frac{x^6}{64}$
- $r = 1$: $\binom{6}{1}\left(\frac{x}{2}\right)^5\left(-\frac{2}{x^2}\right) = 6 \cdot \frac{x^5}{32} \cdot \left(-\frac{2}{x^2}\right) = -6 \cdot \frac{x^3}{16} = -\frac{3x^3}{8}$
- $r = 2$: $\binom{6}{2}\left(\frac{x}{2}\right)^4\left(\frac{4}{x^4}\right) = 15 \cdot \frac{x^4}{16} \cdot \frac{4}{x^4} = 15 \cdot \frac{4}{16} = \frac{15}{4}$
- $r = 3$: $\binom{6}{3}\left(\frac{x}{2}\right)^3\left(-\frac{8}{x^6}\right) = 20 \cdot \frac{x^3}{8} \cdot \left(-\frac{8}{x^6}\right) = 20 \cdot \left(-\frac{1}{x^3}\right) = -\frac{20}{x^3}$
- $r = 4$: $\binom{6}{4}\left(\frac{x}{2}\right)^2\left(\frac{16}{x^8}\right) = 15 \cdot \frac{x^2}{4} \cdot \frac{16}{x^8} = 15 \cdot \frac{4}{x^6} = \frac{60}{x^6}$
- $r = 5$: $\binom{6}{5}\left(\frac{x}{2}\right)^1\left(-\frac{32}{x^{10}}\right) = 6 \cdot \frac{x}{2} \cdot \left(-\frac{32}{x^{10}}\right) = -96 \cdot \frac{1}{x^9} = -\frac{96}{x^9}$
- $r = 6$: $\binom{6}{6}\left(-\frac{2}{x^2}\right)^6 = \frac{64}{x^{12}}$

Answer: $\frac{x^6}{64} - \frac{3x^3}{8} + \frac{15}{4} - \frac{20}{x^3} + \frac{60}{x^6} - \frac{96}{x^9} + \frac{64}{x^{12}}$

(ii) $\left(2a - \frac{x}{a}\right)^7$

Step 1: $a = 2a$, $b = -\frac{x}{a}$, $n = 7$

$$\sum_{r=0}^{7} \binom{7}{r} (2a)^{7-r} \left(-\frac{x}{a}\right)^r$$

Step 2: Calculate each term:

- $r = 0$: $\binom{7}{0}(2a)^7 = 128a^7$
- $r = 1$: $\binom{7}{1}(2a)^6\left(-\frac{x}{a}\right) = 7 \cdot 64a^6 \cdot \left(-\frac{x}{a}\right) = -448a^5x$
- $r = 2$: $\binom{7}{2}(2a)^5\left(\frac{x^2}{a^2}\right) = 21 \cdot 32a^5 \cdot \frac{x^2}{a^2} = 672a^3x^2$
- $r = 3$: $\binom{7}{3}(2a)^4\left(-\frac{x^3}{a^3}\right) = 35 \cdot 16a^4 \cdot \left(-\frac{x^3}{a^3}\right) = -560ax^3$
- $r = 4$: $\binom{7}{4}(2a)^3\left(\frac{x^4}{a^4}\right) = 35 \cdot 8a^3 \cdot \frac{x^4}{a^4} = \frac{280x^4}{a}$
- $r = 5$: $\binom{7}{5}(2a)^2\left(-\frac{x^5}{a^5}\right) = 21 \cdot 4a^2 \cdot \left(-\frac{x^5}{a^5}\right) = -\frac{84x^5}{a^3}$
- $r = 6$: $\binom{7}{6}(2a)^1\left(\frac{x^6}{a^6}\right) = 7 \cdot 2a \cdot \frac{x^6}{a^6} = \frac{14x^6}{a^5}$

- $r = 7$: $\binom{7}{7}\left(-\frac{x^7}{a^7}\right) = -\frac{x^7}{a^7}$

Answer: $128a^7 - 448a^5x + 672a^3x^2 - 560ax^3 + \frac{280x^4}{a} - \frac{84x^5}{a^3} + \frac{14x^6}{a^5} - \frac{x^7}{a^7}$

(iii) $\left(\frac{\mathbf{a}}{\mathbf{x}} - \frac{\mathbf{x}}{\mathbf{a}}\right)^{\mathbf{6}}$

Step 1: $a = \frac{a}{x}$, $b = -\frac{x}{a}$, $n = 6$

$$\sum_{r=0}^{6} \binom{6}{r} \left(\frac{a}{x}\right)^{6-r} \left(-\frac{x}{a}\right)^r$$

Step 2: Calculate each term:

- $r = 0$: $\binom{6}{0}\frac{a^6}{x^6} = \frac{a^6}{x^6}$
- $r = 1$: $\binom{6}{1}\frac{a^5}{x^5}\left(-\frac{x}{a}\right) = -6 \cdot \frac{a^4}{x^4}$
- $r = 2$: $\binom{6}{2}\frac{a^4}{x^4}\left(\frac{x^2}{a^2}\right) = 15 \cdot \frac{a^2}{x^2}$
- $r = 3$: $\binom{6}{3}\frac{a^3}{x^3}\left(-\frac{x^3}{a^3}\right) = 20(-1) = -20$
- $r = 4$: $\binom{6}{4}\frac{a^2}{x^2}\left(\frac{x^4}{a^4}\right) = 15 \cdot \frac{x^2}{a^2}$
- $r = 5$: $\binom{6}{5}\frac{a}{x}\left(-\frac{x^5}{a^5}\right) = -6 \cdot \frac{x^4}{a^4}$
- $r = 6$: $\binom{6}{6}\left(\frac{x^6}{a^6}\right) = \frac{x^6}{a^6}$

Answer: $\frac{a^6}{x^6} - \frac{6a^4}{x^4} + \frac{15a^2}{x^2} - 20 + \frac{15x^2}{a^2} - \frac{6x^4}{a^4} + \frac{x^6}{a^6}$

Question 2: Calculate using binomial theorem

(i) $(\mathbf{0.97})^{\mathbf{3}}$

Step 1: $0.97 = 1 - 0.03$

$()^3 = 1 - 3(0.03) + 3(0.03)^2 - (0.03)^3 = 1 - 0.09 + 3(0.0009) - 0.000027 = 1 - 0.09 + 0.0027 - 0.000027 = 0.912673$

Answer: 0.912673

(ii) $(\mathbf{2.02})^{\mathbf{4}}$

Step 1: $2.02 = 2 + 0.02$

$()^4 = 2^4 + 4(2^3)(0.02) + 6(2^2)(0.02)^2 + 4(2)(0.02)^3 + (0.02)^4 = 16 + 4(8)(0.02) + 6(4)(0.0004) + 8(0.000008) + 0.00000016 = 1$

Answer: 16.64966416

(iii) $(\mathbf{9.98})^{\mathbf{4}}$

Step 1: $9.98 = 10 - 0.02$

$()^4 = 10^4 - 4(10^3)(0.02) + 6(10^2)(0.02)^2 - 4(10)(0.02)^3 + (0.02)^4 = 10000 - 4(1000)(0.02) + 6(100)(0.0004) - 40(0.000008) +$

Answer: 9920.23968016

(iv) $(\mathbf{2.1})^{\mathbf{5}}$

Step 1: $2.1 = 2 + 0.1$

$()^5 = 2^5 + 5(2^4)(0.1) + 10(2^3)(0.1)^2 + 10(2^2)(0.1)^3 + 5(2)(0.1)^4 + (0.1)^5 = 32 + 5(16)(0.1) + 10(8)(0.01) + 10(4)(0.001) + 10($

Answer: 40.84101

Question 3: Expand and simplify

(i) $\left(\mathbf{a}+\sqrt{\mathbf{2}}\mathbf{x}\right)^{\mathbf{4}}+\left(\mathbf{a}-\sqrt{\mathbf{2}}\mathbf{x}\right)^{\mathbf{4}}$

Step 1: Expand using binomial theorem

$(a + \sqrt{2}x)^4 = a^4 + 4a^3(\sqrt{2}x) + 6a^2(2x^2) + 4a(2\sqrt{2}x^3) + 4x^4 = a^4 + 4\sqrt{2}a^3x + 12a^2x^2 + 8\sqrt{2}ax^3 + 4x^4 \big(a - \sqrt{2}x)^4 = a^4 - 4\sqrt{}$

Step 2: Add

$$\big(a + \sqrt{2}x)^4 + (a - \sqrt{2}x)^4 = 2a^4 + 24a^2x^2 + 8x^4$$

Answer: $2a^4 + 24a^2x^2 + 8x^4$

(ii) $\left(\mathbf{2}+\sqrt{\mathbf{3}}\right)^{\mathbf{6}}+\left(\mathbf{2}-\sqrt{\mathbf{3}}\right)^{\mathbf{6}}$

Step 1: Using the pattern $(a + b)^n + (a - b)^n = 2[a^n + \binom{n}{2}a^{n-2}b^2 + \binom{n}{4}a^{n-4}b^4 + \cdots]$

For $n = 6$:

$= 2\left[2^6 + \binom{6}{2}2^4(\sqrt{3})^2 + \binom{6}{4}2^2(\sqrt{3})^4 + \binom{6}{6}(\sqrt{3})^6\right] = 2[64 + 15(16)(3) + 15(4)(9) + 1(27)] = 2[64 + 720 + 540 + 27] = 2[135$

Answer: 2702

Question 4: Expand in ascending power of x

(i) $(\mathbf{2} + \mathbf{x} - \mathbf{x^2})^{\mathbf{4}}$

Step 1: Write $2 + x - x^2 = 2 + (x - x^2)$

$$\Big(x^2))^4 = \sum_{r=0}^{4} \binom{4}{r} 2^{4-r}(x - x^2\Big)^r$$

Step 2: Expand:

- $r = 0$: 16
- $r = 1$: $4 \cdot 8 \cdot (x - x^2) = 32x - 32x^2$
- $r = 2$: $6 \cdot 4 \cdot (x - x^2)^2 = 24(x^2 - 2x^3 + x^4) = 24x^2 - 48x^3 + 24x^4$

- $r = 3$: $4 \cdot 2 \cdot (x - x^2)^3 = 8(x^3 - 3x^4 + 3x^5 - x^6) = 8x^3 - 24x^4 + 24x^5 - 8x^6$
- $r = 4$: $1 \cdot 1 \cdot (x - x^2)^4 = x^4 - 4x^5 + 6x^6 - 4x^7 + x^8$

Step 3: Collect like terms:

- x^0: 16
- x^1: 32
- x^2: $-32 + 24 = -8$
- x^3: $-48 + 8 = -40$
- x^4: $24 - 24 + 1 = 1$
- x^5: $24 - 4 = 20$
- x^6: $-8 + 6 = -2$
- x^7: -4
- x^8: 1

Answer: $16 + 32x - 8x^2 - 40x^3 + x^4 + 20x^5 - 2x^6 - 4x^7 + x^8$

(ii) $\left(\mathbf{1 - x + x^2}\right)^{\mathbf{4}}$

Step 1: $\left(x^2)\right)^4 = \sum_{r=0}^{4} \binom{4}{r}(-x + x^2)^r$

Step 2: Expand:

- $r = 0$: 1
- $r = 1$: $4(-x + x^2) = -4x + 4x^2$
- $r = 2$: $6(-x + x^2)^2 = 6(x^2 - 2x^3 + x^4) = 6x^2 - 12x^3 + 6x^4$
- $r = 3$: $4(-x + x^2)^3 = 4(-x^3 + 3x^4 - 3x^5 + x^6) = -4x^3 + 12x^4 - 12x^5 + 4x^6$
- $r = 4$: $1(-x + x^2)^4 = x^4 - 4x^5 + 6x^6 - 4x^7 + x^8$

Step 3: Collect like terms:

- x^0: 1
- x^1: -4
- x^2: $4 + 6 = 10$
- x^3: $-12 - 4 = -16$
- x^4: $6 + 12 + 1 = 19$
- x^5: $-12 - 4 = -16$
- x^6: $4 + 6 = 10$
- x^7: -4
- x^8: 1

Answer: $1 - 4x + 10x^2 - 16x^3 + 19x^4 - 16x^5 + 10x^6 - 4x^7 + x^8$

Summary Table

1(i)	$\frac{x^6}{64} - \frac{3x^3}{8} + \frac{15}{4} - \frac{20}{x^3} + \frac{60}{x^6} - \frac{96}{x^9} + \frac{64}{x^{12}}$

1(ii)	$128a^7 - 448a^5x + 672a^3x^2 - 560ax^3 + \frac{280x^4}{a} - \frac{84x^5}{a^3} + \frac{14x^6}{a^5} - \frac{x^7}{a^7}$
1(iii)	$\frac{a^6}{x^6} - \frac{6a^4}{x^4} + \frac{15a^2}{x^2} - 20 + \frac{15x^2}{a^2} - \frac{6x^4}{a^4} + \frac{x^6}{a^6}$
2(i)	0.912673
2(ii)	16.64966416
2(iii)	9920.23968016
2(iv)	40.84101
3(i)	$2a^4 + 24a^2x^2 + 8x^4$
3(ii)	2702
4(i)	$16 + 32x - 8x^2 - 40x^3 + x^4 + 20x^5 - 2x^6 - 4x^7 + x^8$
4(ii)	$1 - 4x + 10x^2 - 16x^3 + 19x^4 - 16x^5 + 10x^6 - 4x^7 + x^8$

EXERCISE 8.2 - BINOMIAL THEOREM (Continued)

Question 5: Find the term involving specified power

General Formula

$$T_{r+1} = \binom{n}{r} a^{n-r} b^r$$

(i) $\mathbf{x^4}$ in $(\mathbf{3 - 2x})^{\mathbf{7}}$

Step 1: General term:

$$T_{r+1} = \binom{7}{r}(3)^{7-r}(-2x)^r = \binom{7}{r}3^{7-r}(-2)^r x^r$$

Step 2: Need $r = 4$

$$T_5 = \binom{7}{4}3^3(-2)^4x^4 = 35 \times 27 \times 16x^4 = 15120x^4$$

Answer: $15120x^4$

(ii) $\mathbf{x^2}$ in $(\mathbf{x} - \frac{\mathbf{2}}{\mathbf{x^2}})^{\mathbf{13}}$

Step 1: General term:

$$T_{r+1} = \binom{13}{r}x^{13-r}\left(-\frac{2}{x^2}\right)^r = \binom{13}{r}(-2)^r x^{13-r-2r} = \binom{13}{r}(-2)^r x^{13-3r}$$

Step 2: Need $13 - 3r = 2 \to 3r = 11 \to r = \frac{11}{3}$ (not an integer)

No term contains x^2

Answer: No such term exists

(iii) $\mathbf{a^4}$ in $(\frac{\mathbf{2}}{\mathbf{x}} - \mathbf{a})^{\mathbf{9}}$

Step 1: General term:

$$T_{r+1} = \binom{9}{r}\left(\frac{2}{x}\right)^{9-r}(-a)^r = \binom{9}{r}2^{9-r}(-1)^r a^r x^{-(9-r)}$$

Step 2: Need $r = 4$

$$T_5 = \binom{9}{4}2^5(-1)^4 a^4 x^{-5} = 126 \times 32a^4x^{-5} = 4032\frac{a^4}{x^5}$$

Answer: $4032\frac{a^4}{x^5}$

(iv) $\mathbf{y^3}$ in $(\mathbf{x} - \sqrt{\mathbf{y}})^{11}$

Step 1: General term:

$$T_{r+1} = \binom{11}{r}x^{11-r}(-\sqrt{y})^r = \binom{11}{r}(-1)^r x^{11-r}y^{r/2}$$

Step 2: Need $\frac{r}{2} = 3 \to r = 6$

$$T_7 = \binom{11}{6}(-1)^6x^5y^3 = 462x^5y^3$$

Answer: $462x^5y^3$

Question 6: Find the coefficient of specified term

(i) $\mathbf{x^3}$ in $(\mathbf{x^2} - \frac{\mathbf{3}}{\mathbf{2x}})^{10}$

Step 1: General term:

$$T_{r+1} = \binom{10}{r}(x^2)^{10-r}\left(-\frac{3}{2x}\right)^r = \binom{10}{r}\left(-\frac{3}{2}\right)^r x^{20-2r-r} = \binom{10}{r}\left(-\frac{3}{2}\right)^r x^{20-3r}$$

Step 2: Need $20 - 3r = 3 \to 3r = 17 \to r = \frac{17}{3}$ (not an integer)

No term contains x^3

Answer: No such term exists

(ii) $\mathbf{x^7}$ in $(\mathbf{x^2} - \frac{\mathbf{1}}{\mathbf{x}})^{28}$

Step 1: General term:

$$T_{r+1} = \binom{28}{r}(x^2)^{28-r}\left(-\frac{1}{x}\right)^r = \binom{28}{r}(-1)^r x^{56-2r-r} = \binom{28}{r}(-1)^r x^{56-3r}$$

Step 2: Need $56 - 3r = 7 \to 3r = 49 \to r = \frac{49}{3}$ (not an integer)

No term contains x^7

Answer: No such term exists

Question 7: Find $\mathbf{6^{th}}$ term in $(\mathbf{x^2} - \frac{\mathbf{3}}{\mathbf{2x}})^{10}$

Step 1: 6^{th} term means $r = 5$

$$T_6 = \binom{10}{5}(x^2)^5\left(-\frac{3}{2x}\right)^5 = 252 \cdot x^{10} \cdot \left(-\frac{243}{32x^5}\right) = 252 \cdot \left(-\frac{243}{32}\right)x^5 = -\frac{61236}{32}x^5 = -\frac{15309}{8}x^5$$

Answer: $-\frac{15309}{8}x^5$

Question 8: Find term independent of x

(i) $(\mathbf{x} - \frac{2}{\mathbf{x}})^{10}$

Step 1: General term:

$$T_{r+1} = \binom{10}{r}x^{10-r}\left(-\frac{2}{x}\right)^r = \binom{10}{r}(-2)^r x^{10-2r}$$

Step 2: Need $10 - 2r = 0 \to r = 5$

$$T_6 = \binom{10}{5}(-2)^5 = 252 \times (-32) = -8064$$

Answer: -8064

(ii) $(\sqrt{\mathbf{x}} + \frac{1}{2\mathbf{x}^2})^{10}$

Step 1: General term:

$$T_{r+1} = \binom{10}{r}(\sqrt{x})^{10-r}\left(\frac{1}{2x^2}\right)^r = \binom{10}{r}\frac{1}{2^r}x^{\frac{10-r}{2}-2r} = \binom{10}{r}\frac{1}{2^r}x^{\frac{10-5r}{2}}$$

Step 2: Need $\frac{10-5r}{2} = 0 \to 10 - 5r = 0 \to r = 2$

$$T_3 = \binom{10}{2}\frac{1}{2^2} = 45 \times \frac{1}{4} = \frac{45}{4}$$

Answer: $\frac{45}{4}$

(iii) $\left(\mathbf{1} + \mathbf{x}^2\right)^3 (\mathbf{1} + \frac{1}{\mathbf{x}^2})^4$

Step 1: Expand each:

$$(1+x^2)^3 = 1 + 3x^2 + 3x^4 + x^6 \quad \left(1 + \frac{1}{x^2}\right)^4 = 1 + \frac{4}{x^2} + \frac{6}{x^4} + \frac{4}{x^6} + \frac{1}{x^8}$$

Step 2: Product terms independent of x:

- $1 \times 1 = 1$
- $3x^2 \times \frac{4}{x^2} = 12$
- $3x^4 \times \frac{6}{x^4} = 18$
- $x^6 \times \frac{4}{x^6} = 4$

Step 3: Sum $= 1 + 12 + 18 + 4 = 35$

Answer: 35

Question 9: Determine the middle term

(i) $\left(\frac{1}{x} - \frac{x^2}{2}\right)^{12}$

Step 1: $n = 12$ (even), so one middle term: $r = \frac{n}{2} = 6$

$$T_7 = \binom{12}{6}\left(\frac{1}{x}\right)^6\left(-\frac{x^2}{2}\right)^6 = 924 \cdot \frac{1}{x^6} \cdot \frac{x^{12}}{2^6} = 924 \cdot \frac{x^6}{64} = \frac{231}{16}x^6$$

Answer: $\frac{231}{16}x^6$

(ii) $\left(\frac{3}{2}x - \frac{1}{3x}\right)^{11}$

Step 1: $n = 11$ (odd), so two middle terms: $r = 5$ and $r = 6$

For $r = 5$ (6th term):

$$T_6 = \binom{11}{5}\left(\frac{3x}{2}\right)^6\left(-\frac{1}{3x}\right)^5 = 462 \cdot \frac{3^6x^6}{2^6} \cdot \left(-\frac{1}{3^5x^5}\right) = -462 \cdot \frac{3}{64}x = -\frac{1386}{64}x = -\frac{693}{32}x$$

For $r = 6$ (7th term):

$$T_7 = \binom{11}{6}\left(\frac{3x}{2}\right)^5\left(-\frac{1}{3x}\right)^6 = 462 \cdot \frac{3^5x^5}{2^5} \cdot \frac{1}{3^6x^6} = 462 \cdot \frac{1}{3 \cdot 32x} = \frac{462}{96x} = \frac{77}{16x}$$

Answer: $-\frac{693}{32}x$ and $\frac{77}{16x}$

(iii) $\left(2x - \frac{1}{2x}\right)^{2n+1}$

Step 1: n is odd $(2n + 1)$, so two middle terms: $r = n$ and $r = n + 1$

For $r = n$:

$$T_{n+1} = \binom{2n+1}{n}(2x)^{2n+1-n}\left(-\frac{1}{2x}\right)^n = \binom{2n+1}{n}2^{n+1}x^{n+1} \cdot \left(-\frac{1}{2^nx^n}\right) = \binom{2n+1}{n}(-1)^n2x$$

For $r = n + 1$:

$$T_{n+2} = \binom{2n+1}{n+1}(2x)^{2n+1-(n+1)}\left(-\frac{1}{2x}\right)^{n+1} = \binom{2n+1}{n+1}2^nx^n \cdot \left(-\frac{1}{2^{n+1}x^{n+1}}\right) = \binom{2n+1}{n+1}\frac{(-1)^{n+1}}{2x}$$

Answer: $\binom{2n+1}{n}(-1)^n2x$ and $\binom{2n+1}{n+1}\frac{(-1)^{n+1}}{2x}$

Question 10: Show that $\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n}=2^n-1$

Step 1: By binomial theorem:

$$\Big(1 + 1)^n = \sum_{r=0}^{n}\binom{n}{r} = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n$$

Step 2: Since $\binom{n}{0} = 1$:

$$1 + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$$

Step 3: Therefore:

$$\binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n - 1$$

Therefore, the identity is proven

Summary Table

5(i)	$15120x^4$
5(ii)	No term exists
5(iii)	$4032\frac{a^4}{x^5}$
5(iv)	$462x^5y^3$
6(i)	No term exists
6(ii)	No term exists
7	$-\frac{15309}{8}x^5$
8(i)	-8064
8(ii)	$\frac{45}{4}$
8(iii)	35
9(i)	$\frac{231}{16}x^6$
9(ii)	$-\frac{693}{32}x$ and $\frac{77}{16x}$
9(iii)	$\binom{2n+1}{n}(-1)^n 2x$ and $\binom{2n+1}{n+1}\frac{(-1)^{n+1}}{2x}$
10	Proven