EXERCISE 8.3 - BINOMIAL EXPANSION (Generalized)

Complete Solutions

Question 1: Expand up to 4 terms

(i) $(1+x)^{-1/3}$

Step 1: Use binomial series: $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$

- Here $n = -\frac{1}{3}$

Step 2: Expand:

$$(1+x)^{-1/3} = 1 + \left(-\frac{1}{3}\right)x + \frac{\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)}{2}x^2 + \frac{\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)\left(-\frac{7}{3}\right)}{6}x^3 + \cdots = 1 - \frac{x}{3} + \frac{2}{9}x^2 - \frac{14}{81}x^3 + \cdots$$

Validity: $|x| < 1$

Answer: $1 - \frac{x}{3} + \frac{2}{9}x^2 - \frac{14}{81}x^3 + \cdots$

(ii) $(4-3x)^{1/2}$

Step 1: Rewrite: $(4-3x)^{1/2} = \sqrt{4}\left(1 - \frac{3x}{4}\right)^{1/2} = 2\left(1 - \frac{3x}{4}\right)^{1/2}$

Step 2: Expand with $n = \frac{1}{2}$:

$$2\left[1 + \frac{1}{2}\left(-\frac{3x}{4}\right) + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2}\left(-\frac{3x}{4}\right)^2 + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{6}\left(-\frac{3x}{4}\right)^3 + \cdots\right]$$

Step 3: Simplify:

- Term 0: 2
- Term 1: $2 \cdot \frac{1}{2}\left(-\frac{3x}{4}\right) = -\frac{3x}{4}$
- Term 2: $2 \cdot \frac{-1}{8}\left(\frac{9x^2}{16}\right) = -\frac{9x^2}{64}$
- Term 3: $2 \cdot \frac{1}{16}\left(-\frac{27x^3}{64}\right) = -\frac{27x^3}{512}$

Answer: $2 - \frac{3}{4}x - \frac{9}{64}x^2 - \frac{27}{512}x^3 + \cdots$

Validity: $\left|\frac{3x}{4}\right| < 1 \rightarrow |x| < \frac{4}{3}$

(iii) $\frac{(1-x)^{-1}}{(1+x)^2}$

Step 1: $\frac{1}{1-x} \cdot (1+x)^{-2} = (1-x)^{-1}(1+x)^{-2}$

Step 2: Expand each:

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots (1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \cdots$$

Step 3: Multiply up to x^3:

$$(1 + x + x^2 + x^3)(1 - 2x + 3x^2 - 4x^3)$$

- x^0: 1

- x^1: $-2+1=-1$
- x^2: $3-2+1=2$
- x^3: $-4+3-2+1=-2$

Answer: $1-x+2x^2-2x^3+\cdots$

Validity: $|x|<1$

(iv) $\frac{\sqrt{1+2x}}{1-x}$

Step 1: $(1+2x)^{1/2}(1-x)^{-1}$

Step 2: Expand:

$$(1+2x)^{1/2}=1+\frac{1}{2}(2x)+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2}(2x)^2+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{6}(2x)^3+\cdots=1+x-\frac{x^2}{2}+\frac{x^3}{2}+\cdots(1-x)^{-1}=1+x+x^2+x^3+\cdots$$

Step 3: Multiply:

$$\left(1+x-\frac{x^2}{2}+\frac{x^3}{2})(1+x+x^2+x^3\right)$$

- x^0: 1
- x^1: $1+1=2$
- x^2: $1-\frac{1}{2}+1=\frac{3}{2}$
- x^3: $1-\frac{1}{2}+1+\frac{1}{2}=2$

Answer: $1+2x+\frac{3}{2}x^2+2x^3+\cdots$

Validity: $|2x|<1$ and $|x|<1\to|x|<\frac{1}{2}$

Question 2: Coefficient of $\mathbf{x^n}$

(i) $\frac{1+x^2}{(1+x)^2}$

Step 1: $\frac{1+x^2}{(1+x)^2}=(1+x^2)(1+x)^{-2}$

Step 2: $(1+x)^{-2}=\sum_{r=0}^{\infty}(-1)^r(r+1)x^r$

Step 3: Coefficient of x^n:

- From $1\cdot(1+x)^{-2}$: coefficient $=(-1)^n(n+1)$
- From $x^2\cdot(1+x)^{-2}$: coefficient of $x^{n-2}=(-1)^{n-2}(n-1)=(-1)^n(n-1)$

Step 4: Total $=(-1)^n(n+1)+(-1)^n(n-1)=(-1)^n(2n)$

Answer: $2n(-1)^n$

(ii) $\frac{(1+x)^2}{(1-x)^2}$

Step 1: $(1+x)^2(1-x)^{-2}$

Step 2: $(1-x)^{-2}=\sum_{r=0}^{\infty}(r+1)x^r$

Step 3: $(1+x)^2=1+2x+x^2$

Step 4: Coefficient of x^n:

- From 1: $n+1$
- From $2x$: $2n$
- From x^2: $n-1$ (for $n \geq 1$)

Step 5: Total $= (n+1) + 2n + (n-1) = 4n$ (for $n \geq 1$)
For $n = 0$: coefficient $= 1$

Answer: Coefficient of x^n is $4n$ for $n \geq 1$, and 1 for $n = 0$

Question 3: Approximations (neglecting $\mathbf{x^2}$ and higher)

(i) $\frac{1-x}{\sqrt{1+x}} \approx 1 - \frac{3}{2}x$

Step 1: $(1+x)^{-1/2} = 1 - \frac{x}{2} + \cdots$

Step 2: $(1-x)(1+x)^{-1/2} = (1-x)\left(1 - \frac{x}{2}\right) = 1 - \frac{x}{2} - x + \frac{x^2}{2} \approx 1 - \frac{3}{2}x$

Therefore, $\frac{1-x}{\sqrt{1+x}} \approx 1 - \frac{3}{2}x$

(ii) $\frac{\sqrt{1+2x}}{\sqrt{1-x}} \approx 1 + \frac{3}{2}x$

Step 1: $(1+2x)^{1/2} = 1 + x + \cdots$
Step 2: $(1-x)^{-1/2} = 1 + \frac{x}{2} + \cdots$

Step 3: $(1+x)\left(1 + \frac{x}{2}\right) = 1 + \frac{x}{2} + x = 1 + \frac{3}{2}x$

Therefore, $\frac{\sqrt{1+2x}}{\sqrt{1-x}} \approx 1 + \frac{3}{2}x$

(iii) $\frac{(9+7x)^{1/2} - (16+3x)^{1/4}}{4+5x} \approx \frac{1}{4} - \frac{17}{384}x$

Step 1: $(9+7x)^{1/2} = 3\left(1 + \frac{7x}{9}\right)^{1/2} = 3\left(1 + \frac{7x}{18}\right) = 3 + \frac{7x}{6}$

Step 2: $(16+3x)^{1/4} = 2\left(1 + \frac{3x}{16}\right)^{1/4} = 2\left(1 + \frac{3x}{64}\right) = 2 + \frac{3x}{32}$

Step 3: Numerator: $\left(3 + \frac{7x}{6}\right) - \left(2 + \frac{3x}{32}\right) = 1 + x\left(\frac{7}{6} - \frac{3}{32}\right) = 1 + x\left(\frac{112-9}{96}\right) = 1 + \frac{103x}{96}$

Step 4: Denominator: $(4+5x)^{-1} = \frac{1}{4}\left(1 + \frac{5x}{4}\right)^{-1} = \frac{1}{4}\left(1 - \frac{5x}{4}\right) = \frac{1}{4} - \frac{5x}{16}$

Step 5: Multiply: $\left(1 + \frac{103x}{96}\right)\left(\frac{1}{4} - \frac{5x}{16}\right) = \frac{1}{4} + x\left(\frac{103}{384} - \frac{5}{64}\right)$

$$= \frac{1}{4} + x\left(\frac{103}{384} - \frac{30}{384}\right) = \frac{1}{4} + \frac{73}{384}x$$

This doesn't match the stated answer. Let's recheck the numerator expansion carefully.

Step 1 (re-evaluated): $(9+7x)^{1/2} = 3\left(1 + \frac{7x}{9}\right)^{1/2}$

- $= 3\left[1 + \frac{1}{2}\left(\frac{7x}{9}\right) - \frac{1}{8}\left(\frac{7x}{9}\right)^2 + \cdots\right]$
- Up to first order: $3 + \frac{7x}{6}$

Step 2: $(16+3x)^{1/4} = 2\left(1 + \frac{3x}{16}\right)^{1/4}$

- $= 2\left[1 + \frac{1}{4}\left(\frac{3x}{16}\right)\right] = 2 + \frac{3x}{32}$

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Step 3: Numerator: $1+\frac{7x}{6}-\frac{3x}{32}=1+\frac{112x-9x}{96}=1+\frac{103x}{96}$

Step 4: Denominator: $\frac{1}{4+5x}=\frac{1}{4}\left(1+\frac{5x}{4}\right)^{-1}=\frac{1}{4}-\frac{5x}{16}$

Step 5: Product: $\frac{1}{4}+x\left(\frac{103}{384}-\frac{5}{64}\right)=\frac{1}{4}+\frac{103-30}{384}x=\frac{1}{4}+\frac{73}{384}x$

The stated answer $\frac{1}{4}-\frac{17}{384}x$ seems to have a different value.

(iv) $\frac{\sqrt{4+x}}{(1-x)^2}\approx 2+\frac{25}{4}x$

Step 1: $\sqrt{4+x}=2\left(1+\frac{x}{4}\right)^{1/2}=2\left(1+\frac{x}{8}\right)=2+\frac{x}{4}$

Step 2: $(1-x)^{-2}=1+2x+\cdots$

Step 3: $\left(2+\frac{x}{4}\right)(1+2x)=2+4x+\frac{x}{4}=2+\frac{17}{4}x$

This doesn't match $2+\frac{25}{4}x$.

Question 4: Approximations (neglecting x^3 and higher)

(i) $\sqrt{1-x}-2x^2\approx 1-\frac{1}{2}x-\frac{9}{8}x^2$

Step 1: $\sqrt{1-x}=(1-x)^{1/2}=1-\frac{x}{2}-\frac{x^2}{8}-\frac{x^3}{16}-\cdots$

Step 2: $\sqrt{1-x}-2x^2=1-\frac{x}{2}-\frac{x^2}{8}-2x^2=1-\frac{x}{2}-\frac{17}{8}x^2$

The stated answer has $-\frac{9}{8}x^2$, not $-\frac{17}{8}x^2$.

(ii) $\frac{1+x}{\sqrt{1-x}}\approx 1+x+\frac{1}{2}x^2$

Step 1: $\left(1-x\right)^{-1/2}=1+\frac{x}{2}+\frac{3x^2}{8}+\cdots$

Step 2: $(1+x)\left(1+\frac{x}{2}+\frac{3x^2}{8}\right)=1+\frac{x}{2}+x+\frac{3x^2}{8}+\frac{x^2}{2}=1+\frac{3x}{2}+\frac{7x^2}{8}$

The stated answer has $1+x+\frac{x^2}{2}$.

Question 5: If x is very nearly equal to 1, prove $px^p-qx^q\approx(p-q)x^{p+q}$

Step 1: Let $x=1+h$ where h is very small
Step 2: $x^p=(1+h)^p=1+ph$
Step 3: $x^q=(1+h)^q=1+qh$

Step 4: $px^p-qx^q=p(1+ph)-q(1+qh)=(p-q)+(p^2-q^2)h$

$$=(p-q)[1+(p+q)h]=(p-q)(1+h)^{p+q}\approx(p-q)x^{p+q}$$

Therefore, $px^p-qx^q\approx(p-q)x^{p+q}$

Question 6: Identify series as binomial expansion

$$\frac{1}{1}-\frac{1}{2}\left(\frac{1}{4}\right)+\frac{1}{2}\left(\frac{1}{4}\right)^2-\frac{1}{3}\left(\frac{1}{4}\right)^3+\frac{1}{3}\left(\frac{1}{4}\right)^4+\cdots$$

Step 1: Let $x=\frac{1}{4}$
Step 2: Series: $1-\frac{x}{2}+\frac{x^2}{2}-\frac{x^3}{3}+\frac{x^4}{3}-\cdots$

This looks like $(1+x)^{-1}$ with some modifications.

Step 3: $(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - \cdots$

Step 4: $\frac{1}{2}(1+x)^{-1} + \frac{1}{2}(1-x)^{-1}$:

$$= \frac{1}{2}[(1 - x + x^2 - x^3 + \cdots) + (1 + x + x^2 + x^3 + \cdots)] = \frac{1}{2}[2 + 2x^2 + 2x^4 + \cdots] = 1 + x^2 + x^4 + \cdots$$

This doesn't match. The series appears to be $(1+x)^{-1/2}(1-x)^{-1/2}$ or similar.

Question 7: Show that $\frac{1}{4}+\frac{1}{4}\cdot\frac{1}{8}+\frac{1}{4}\cdot\frac{1}{8}\cdot\frac{1}{12}+\cdots=\sqrt{2}$

Step 1: Consider $\left(1+x\right)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots$

Step 2: For $x = 1$, we need $1 + n + \frac{n(n-1)}{2} + \cdots$

Step 3: The terms suggest $n = \frac{1}{2}$ and $x = 1$:

$$\Big(1+1)^{1/2} = \sqrt{2} = 1 + \frac{1}{2} + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2} + \cdots = 1 + \frac{1}{2} - \frac{1}{8} + \cdots$$

This is not matching the given series.

The given series appears to be related to $(1-x)^{-1/2}$ with $x = \frac{1}{2}$.

Summary Table

1(i)	$1 - \frac{x}{3} + \frac{2}{9}x^2 - \frac{14}{81}x^3 + \cdots$
1(ii)	$2 - \frac{3}{4}x - \frac{9}{64}x^2 - \frac{27}{512}x^3 + \cdots$
1(iii)	$1 - x + 2x^2 - 2x^3 + \cdots$
1(iv)	$1 + 2x + \frac{3}{2}x^2 + 2x^3 + \cdots$
2(i)	$2n(-1)^n$
2(ii)	$4n$ (for $n \geq 1$), 1 (for $n = 0$)
3(i)	Proven
3(ii)	Proven
4(ii)	$1 + \frac{3}{2}x + \frac{7}{8}x^2$ (discrepancy)
5	Proven

EXERCISE 8.3 - BINOMIAL EXPANSION (Continued)

Question 8: If $y = \frac{1}{3} + \frac{1}{2!}\left(\frac{1}{3}\right)^2 + \frac{1}{3!}\left(\frac{1}{3}\right)^3 + \cdots$, prove that $y^2 + 2y - 2 = 0$

Step 1: Observe the series pattern:

$$y = \sum_{n=1}^{\infty} \frac{1}{n!}\left(\frac{1}{3}\right)^n$$

Step 2: This is the Taylor series for $e^x - 1$ with $x = \frac{1}{3}$:

$$e^{1/3} = 1 + \frac{1}{3} + \frac{1}{2!}\left(\frac{1}{3}\right)^2 + \frac{1}{3!}\left(\frac{1}{3}\right)^3 + \cdots y = e^{1/3} - 1$$

Step 3: Therefore $e^{1/3} = y + 1$

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Step 4: Cube both sides:

$$e^1 = (y+1)^3 e = (y+1)^3$$

Step 5: But this doesn't give $y^2 + 2y - 2 = 0$. Let's re-examine.

Step 6: Wait, the series is:

$$y = \frac{1}{3} + \frac{1}{2!}\left(\frac{1}{3}\right)^2 + \frac{1}{3!}\left(\frac{1}{3}\right)^3 + \cdots = \left(1 + \frac{1}{3} + \frac{1}{2!}\left(\frac{1}{3}\right)^2 + \cdots\right) - 1 = e^{1/3} - 1$$

Step 7: So $y + 1 = e^{1/3}$

Step 8: Cubing both sides:

$$(y+1)^3 = e \;\; y^3 + 3y^2 + 3y + 1 = e$$

Step 9: Since $e \approx 2.718$, this doesn't directly give the quadratic. However, if we note that $e^{1/3}$ is approximately $1 + \frac{1}{3} + \frac{1}{2!} \cdot \frac{1}{9} + \cdots$, the given equation $y^2 + 2y - 2 = 0$ suggests $y = \sqrt{3} - 1$ (positive root).

Check: $(\sqrt{3} - 1)^2 + 2(\sqrt{3} - 1) - 2 = 3 - 2\sqrt{3} + 1 + 2\sqrt{3} - 2 - 2 = 0$

Step 10: So $y = \sqrt{3} - 1 = e^{1/3} - 1$ implies $e^{1/3} = \sqrt{3}$, which means $e = 3\sqrt{3}$, which is approximately $3 \times 1.732 = 5.196$, not 2.718.

The problem statement may be miswritten. If the series were:

$$y = \frac{1}{2} + \frac{1}{2!}\left(\frac{1}{2}\right)^2 + \frac{1}{3!}\left(\frac{1}{2}\right)^3 + \cdots = e^{1/2} - 1$$

Then $y = \sqrt{e} - 1$, giving $(y+1)^2 = e$, which still doesn't give the quadratic.

Answer: The given equation $y^2 + 2y - 2 = 0$ does not follow from the series as written.

Question 9: If $\mathbf{2y = \frac{1}{2^2} + \frac{1}{2!}\left(\frac{1}{2}\right)^2 + \frac{1}{3!}\left(\frac{1}{2}\right)^3 + \cdots}$, prove that $\mathbf{4y^2 + 4y - 1 = 0}$

Step 1: The series:

$$2y = \sum_{n=1}^{\infty} \frac{1}{n!}\left(\frac{1}{2}\right)^n = e^{1/2} - 1$$

Step 2: So $2y + 1 = e^{1/2}$

Step 3: Squaring both sides:

$$(2y+1)^2 = e \;\; 4y^2 + 4y + 1 = e$$

Step 4: To get $4y^2 + 4y - 1 = 0$, we need $e = 2$, which is approximately correct but not exact.

The problem statement likely contains an error in the series or the target equation.

Question 10: Show that the coefficient of $\mathbf{x^r}$ in $\mathbf{\frac{x}{(1-px)(1-qx)}}$ is $\mathbf{\frac{p^r - q^r}{p - q}}$

Step 1: Decompose into partial fractions:

$$\frac{x}{(1-px)(1-qx)} = \frac{A}{1-px} + \frac{B}{1-qx}$$

Step 2: Find A and B:

$$x = A(1-qx) + B(1-px)x = (A+B) - (Aq+Bp)x$$

Comparing coefficients:

- Constant: $A + B = 0 \to B = -A$
- x: $1 = -(Aq + Bp) = -(Aq - Ap) = A(p-q)$
- $A = \frac{1}{p-q}$, $B = -\frac{1}{p-q}$

Step 3: Therefore:

$$\frac{x}{(1-px)(1-qx)} = \frac{1}{p-q}\left(\frac{1}{1-px} - \frac{1}{1-qx}\right)$$

Step 4: Expand using binomial series:

$$\frac{1}{1-px} = \sum_{r=0}^{\infty}(px)^r = \sum_{r=0}^{\infty} p^r x^r \frac{1}{1-qx} = \sum_{r=0}^{\infty} q^r x^r$$

Step 5: Coefficient of x^r:

$$\frac{1}{p-q}(p^r - q^r) = \frac{p^r - q^r}{p-q}$$

Therefore, the coefficient of x^r is $\frac{p^r - q^r}{p-q}$

Summary Table

8	Problem statement appears inconsistent
9	Problem statement appears inconsistent
10	Proven

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