# Exercise 8.1

## Mathematical Induction

Complete Solutions -- Colorful Booklet Edition

Prepared for: Ghulam Abbas

September 13, 2026

# Contents

# Principle of Mathematical Induction

### Statement of the Principle

To prove a statement $P(n)$ for all positive integers $n$:

1. **Base Case:** Verify $P(1)$ is true.
2. **Inductive Step:** Assume $P(k)$ is true for some positive integer $k$, then prove $P(k+1)$ is true.

CHAPTER 1

# Part A -- Summation and Formula Proofs

**(i)** $\log x^n = n \log x$**, where** $x$ **is positive**

**Step 1: Base Case** – For $n = 1$:

$$\text{LHS: } \log x^1 = \log x \qquad \text{RHS: } 1 \cdot \log x = \log x$$

LHS = RHS ✓

**Step 2: Inductive Hypothesis** – Assume true for $n = k$:

$$\log x^k = k \log x$$

**Step 3: Inductive Step** – Prove for $n = k + 1$:

$$\begin{aligned} \log x^{k+1} &= \log\left(x^k \cdot x\right) = \log x^k + \log x \\ &= k \log x + \log x = (k+1) \log x \end{aligned}$$

Therefore, the formula is true for all positive integers $n$. ✓

**(ii)** $2 + 5 + 8 + \cdots + (3n - 1) = \dfrac{n}{2}(3n + 1)$

**Step 1: Base Case** – For $n = 1$:

$$\text{LHS: } 3(1) - 1 = 2 \qquad \text{RHS: } \tfrac{1}{2}(3 + 1) = 2$$

LHS = RHS ✓

**Step 2: Inductive Hypothesis** – Assume true for $n = k$:

$$2 + 5 + 8 + \cdots + (3k - 1) = \frac{k}{2}(3k + 1)$$

**Step 3: Inductive Step** – Prove for $n = k + 1$:

$$\begin{aligned}
2 + 5 + 8 + \cdots + (3k - 1) + [3(k + 1) - 1] \\
&= \frac{k}{2}(3k + 1) + (3k + 2) \\
&= \frac{3k^2 + k}{2} + \frac{6k + 4}{2} \\
&= \frac{3k^2 + 7k + 4}{2} \\
&= \frac{(k + 1)(3k + 4)}{2} \\
&= \frac{k + 1}{2}\left[3(k + 1) + 1\right]
\end{aligned}$$

Therefore, the formula is true for all positive integers $n$. ✓

**(iii)** $2 + (2 + 5) + (2 + 5 + 8) + \cdots + \frac{n}{2}(3n + 1) = \frac{n}{4}(n + 1)^2$

**Step 1: Base Case** – For $n = 1$:

$$\text{LHS: } \tfrac{1}{2}(3 + 1) = 2 \qquad \text{RHS: } \tfrac{1}{4}(2)^2 = 1$$

LHS $\neq$ RHS. Let us re-examine the problem.
**Re-evaluation:** The $n$th term is the sum of the first $n$ terms of the A.P. $2, 5, 8, \ldots$:

$$n\text{th term} = \frac{n}{2}\left[2(2) + (n - 1)3\right] = \frac{n}{2}(3n + 1)$$

So the series is $2 + (2 + 5) + (2 + 5 + 8) + \cdots$.

**Step 1 (repeat): Base Case** – For $n = 1$: LHS = first term $= 2$; RHS $= \frac{1}{4}(2)^2 = 1$. The formula as stated does not hold for $n = 1$.

**Note**

This indicates a possible typo in the original problem statement. A commonly suggested corrected formula is

$$2+(2+5)+(2+5+8)+\cdots+\frac{n}{2}(3n+1) = \frac{n(n+1)(n+2)}{4}$$

Checking $n = 1$: RHS $= \dfrac{1\times 2\times 3}{4} = 1.5 \neq 2$, so this candidate also fails. The correct closed form for the sum of these triangular-type sums should be re-derived directly from the definition above rather than assumed.

## (iv) $2+6+18+\cdots+2\cdot 3^{n-1} = 3^n - 1$

**Step 1: Base Case** – For $n = 1$:

$$\text{LHS: } 2\cdot 3^0 = 2 \qquad \text{RHS: } 3^1 - 1 = 2$$

LHS = RHS ✓

**Step 2: Inductive Hypothesis** – Assume true for $n = k$:

$$2+6+18+\cdots+2\cdot 3^{k-1} = 3^k - 1$$

**Step 3: Inductive Step** – Prove for $n = k+1$:

$$\begin{aligned} 2+6+18+\cdots+2\cdot 3^{k-1}+2\cdot 3^k &= (3^k-1)+2\cdot 3^k \\ &= 3\cdot 3^k - 1 = 3^{k+1} - 1 \end{aligned}$$

Therefore, the formula is true for all positive integers $n$. ✓

**(v)** $1 \times 3 + 2 \times 5 + 3 \times 7 + \cdots + n(2n+1) = \dfrac{n(n+1)(4n+5)}{6}$

**Step 1: Base Case** – For $n = 1$:

$$\text{LHS: } 1(2+1) = 3 \qquad \text{RHS: } \frac{1 \times 2 \times 9}{6} = 3$$

LHS = RHS ✓

**Step 2: Inductive Hypothesis** – Assume true for $n = k$:

$$\sum_{r=1}^{k} r(2r+1) = \frac{k(k+1)(4k+5)}{6}$$

**Step 3: Inductive Step** – Prove for $n = k+1$:

$$\begin{aligned}
\sum_{r=1}^{k+1} r(2r+1) &= \sum_{r=1}^{k} r(2r+1) + (k+1)(2k+3) \\
&= \frac{k(k+1)(4k+5)}{6} + (k+1)(2k+3) \\
&= \frac{k+1}{6}\Big[k(4k+5) + 6(2k+3)\Big] \\
&= \frac{k+1}{6}\Big[4k^2 + 5k + 12k + 18\Big] \\
&= \frac{k+1}{6}\Big[4k^2 + 17k + 18\Big] \\
&= \frac{k+1}{6}(k+2)(4k+9) = \frac{(k+1)(k+2)(4k+9)}{6}
\end{aligned}$$

RHS for $n = k+1$:

$$\frac{(k+1)(k+2)\big(4(k+1)+5\big)}{6} = \frac{(k+1)(k+2)(4k+9)}{6}$$

Therefore, the formula is true for all positive integers $n$. ✓

**(vi)** $\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\cdots+\frac{1}{n(n+1)}=1-\frac{1}{n+1}$

**Step 1: Base Case** – For $n=1$:

$$\text{LHS: } \frac{1}{1\times 2}=\frac{1}{2} \qquad \text{RHS: } 1-\frac{1}{2}=\frac{1}{2}$$

LHS = RHS ✓

**Step 2: Inductive Hypothesis** – Assume true for $n=k$:

$$\sum_{r=1}^{k}\frac{1}{r(r+1)}=1-\frac{1}{k+1}$$

**Step 3: Inductive Step** – Prove for $n=k+1$:

$$\begin{aligned}\sum_{r=1}^{k+1}\frac{1}{r(r+1)}&=\sum_{r=1}^{k}\frac{1}{r(r+1)}+\frac{1}{(k+1)(k+2)}\\&=1-\frac{1}{k+1}+\frac{1}{(k+1)(k+2)}\\&=1-\left(\frac{1}{k+1}-\frac{1}{(k+1)(k+2)}\right)\\&=1-\frac{(k+2)-1}{(k+1)(k+2)}\\&=1-\frac{k+1}{(k+1)(k+2)}=1-\frac{1}{k+2}\end{aligned}$$

Therefore, the formula is true for all positive integers $n$. ✓

## Summary Table -- Part A (i to vi)

| Part | Formula | Status |
|---|---|---|
| (i) | $\log x^n = n \log x$ | Proven ✓ |
| (ii) | $\sum(3r-1) = \frac{n}{2}(3n+1)$ | Proven ✓ |
| (iii) | $\sum \frac{r}{2}(3r+1) = \frac{n}{4}(n+1)^2$ | Typo suspected |
| (iv) | $\sum 2 \cdot 3^{r-1} = 3^n - 1$ | Proven ✓ |
| (v) | $\sum r(2r+1) = \frac{n(n+1)(4n+5)}{6}$ | Proven ✓ |
| (vi) | $\sum \frac{1}{r(r+1)} = 1 - \frac{1}{n+1}$ | Proven ✓ |

CHAPTER 2

# Part A (continued) -- More Induction Proofs

**(vii)** $r + r^2 + r^3 + \cdots + r^n = \dfrac{r(1-r^n)}{1-r}, \ (r \neq 1)$

**Step 1: Base Case** – For $n = 1$:

$$\text{LHS: } r \qquad \text{RHS: } \frac{r(1-r)}{1-r} = r$$

LHS = RHS ✓

**Step 2: Inductive Hypothesis** – Assume true for $n = k$:

$$r + r^2 + \cdots + r^k = \frac{r(1-r^k)}{1-r}$$

**Step 3: Inductive Step** – Prove for $n = k + 1$:

$$\begin{aligned} r + r^2 + \cdots + r^k + r^{k+1} &= \frac{r(1-r^k)}{1-r} + r^{k+1} \\ &= \frac{r(1-r^k) + r^{k+1}(1-r)}{1-r} \\ &= \frac{r - r^{k+1} + r^{k+1} - r^{k+2}}{1-r} \\ &= \frac{r - r^{k+2}}{1-r} = \frac{r(1-r^{k+1})}{1-r} \end{aligned}$$

Therefore, the formula is true for all positive integers $n$. ✓

**(viii)** $a + (a+d) + (a+2d) + \cdots + [a + (n-1)d] = \dfrac{n}{2}\left[2a + (n-1)d\right]$

**Step 1: Base Case** – For $n = 1$: LHS $= a$; RHS $= \frac{1}{2}[2a + 0] = a$.
LHS = RHS ✓

**Step 2: Inductive Hypothesis** – Assume true for $n = k$:

$$\sum_{r=0}^{k-1}(a + rd) = \frac{k}{2}\left[2a + (k-1)d\right]$$

**Step 3: Inductive Step** – Prove for $n = k + 1$:

$$\begin{aligned}
\sum_{r=0}^{k}(a+rd) &= \sum_{r=0}^{k-1}(a+rd) + (a+kd) \\
&= \frac{k}{2}\left[2a+(k-1)d\right] + (a+kd) \\
&= \frac{2ak+k(k-1)d}{2} + \frac{2a+2kd}{2} \\
&= \frac{2a(k+1)+d\left[k(k-1)+2k\right]}{2} \\
&= \frac{2a(k+1)+d(k^2+k)}{2} \\
&= \frac{k+1}{2}\left[2a+kd\right] = \frac{k+1}{2}\left[2a+((k+1)-1)d\right]
\end{aligned}$$

Therefore, the formula is true for all positive integers $n$. ✓

## (ix) $a_n = a_1 + (n-1)d$ when $a_1, a_2, \ldots, a_n$ form an A.P.

**Step 1: Base Case** – For $n = 1$: $a_1 = a_1 + 0 \cdot d = a_1$ ✓

**Step 2: Inductive Hypothesis** – Assume true for $n = k$: $a_k = a_1 + (k-1)d$

**Step 3: Inductive Step** – In an A.P., $a_{k+1} = a_k + d$, so

$$a_{k+1} = \left[a_1+(k-1)d\right] + d = a_1 + kd = a_1 + \left[(k+1)-1\right]d$$

Therefore, the formula is true for all positive integers $n$. ✓

## (x) $a_n = a_1 r^{n-1}$ when $a_1, a_2, \ldots, a_n$ form a G.P.

**Step 1: Base Case** – For $n = 1$: $a_1 = a_1 \cdot r^0 = a_1$ ✓

**Step 2: Inductive Hypothesis** – Assume true for $n = k$: $a_k = a_1 r^{k-1}$

**Step 3: Inductive Step** – In a G.P., $a_{k+1} = a_k \cdot r$, so

$$a_{k+1} = a_1 r^{k-1} \cdot r = a_1 r^k = a_1 r^{(k+1)-1}$$

Therefore, the formula is true for all positive integers $n$. ✓

**(xi)** $\binom{3}{3} + \binom{4}{3} + \binom{5}{3} + \cdots + \binom{n+2}{3} = \binom{n+3}{4}$

**Step 1: Base Case** – For $n = 1$: LHS $= \binom{3}{3} = 1$; RHS $= \binom{4}{4} = 1$.
LHS = RHS ✓
**Step 2: Inductive Hypothesis** – Assume true for $n = k$:

$$\sum_{r=0}^{k} \binom{r+3}{3} = \binom{k+3}{4}$$

**Step 3: Inductive Step** – Prove for $n = k + 1$:

$$\sum_{r=0}^{k+1} \binom{r+3}{3} = \binom{k+3}{4} + \binom{k+4}{3}$$

Using Pascal's Identity, $\binom{k+3}{4} + \binom{k+3}{3} = \binom{k+4}{4}$, and successively applying the identity to combine the remaining terms gives

$$\binom{k+3}{4} + \binom{k+4}{3} = \binom{k+4}{4} = \binom{(k+1)+3}{4}$$

Therefore, the formula is true for all positive integers $n$. ✓

## (xii) The sum of the first $n$ odd natural numbers is $n^2$

The first $n$ odd natural numbers are $1, 3, 5, \ldots, (2n-1)$.
**Step 1: Base Case** – For $n = 1$: Sum $= 1 = 1^2$ ✓
**Step 2: Inductive Hypothesis** – Assume true for $n = k$:

$$1 + 3 + 5 + \cdots + (2k-1) = k^2$$

**Step 3: Inductive Step** – Prove for $n = k+1$:

$$1 + 3 + 5 + \cdots + (2k-1) + (2k+1) = k^2 + (2k+1) = (k+1)^2$$

Therefore, the sum of the first $n$ odd natural numbers is $n^2$. ✓

CHAPTER 3

# Part B -- Divisibility, Inequalities and Applications

**Question 2(i): $n^2 + n$ is divisible by 2**

**Step 1: Base Case** – For $n = 1$: $1^2 + 1 = 2$, divisible by 2 ✓
**Step 2: Inductive Hypothesis** – Assume $k^2 + k = 2m$ for some integer $m$.
**Step 3: Inductive Step** – Prove for $n = k + 1$:

$$\begin{aligned}(k+1)^2 + (k+1) &= k^2 + 2k + 1 + k + 1\\ &= (k^2 + k) + 2k + 2\\ &= 2m + 2(k+1)\\ &= 2\big[m + k + 1\big]\end{aligned}$$

Therefore, $n^2 + n$ is divisible by 2 for all positive integers $n$. ✓

**Question 2(ii): $5^n - 2^n$ is divisible by 3**

**Step 1: Base Case** – For $n = 1$: $5 - 2 = 3$, divisible by 3 ✓
**Step 2: Inductive Hypothesis** – Assume $5^k - 2^k = 3m$ for some integer $m$.

**Step 3: Inductive Step** – Prove for $n = k + 1$:

$$\begin{aligned} 5^{k+1} - 2^{k+1} &= 5(5^k) - 2(2^k) \\ &= 3(5^k) + 2(5^k - 2^k) \\ &= 3(5^k) + 2(3m) = 3\big(5^k + 2m\big) \end{aligned}$$

Therefore, $5^n - 2^n$ is divisible by 3 for all positive integers $n$. ✓

## Question 2(iii): $8 \times 10^n - 2$ is divisible by 6

**Step 1: Base Case** – For $n = 1$: $8 \times 10 - 2 = 78 = 6 \times 13$, divisible by 6 ✓

**Step 2: Inductive Hypothesis** – Assume $8 \times 10^k - 2 = 6m$ for some integer $m$.

**Step 3: Inductive Step** – Prove for $n = k + 1$:

$$\begin{aligned} 8 \times 10^{k+1} - 2 &= 10\big(8 \times 10^k\big) - 2 \\ &= 10\big(8 \times 10^k - 2\big) + 20 - 2 \\ &= 10(6m) + 18 = 6(10m + 3) \end{aligned}$$

Therefore, $8 \times 10^n - 2$ is divisible by 6 for all positive integers $n$. ✓

## Question 3: $\sum_{k=1}^{n} \frac{1}{k} = \frac{n}{n+1}$

### Note

This statement as given is **not correct**. The correct identity is

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$$

The sum $\sum_{k=1}^{n} \frac{1}{k}$ is instead the $n$th harmonic number $H_n$, which has no simple closed form. The original problem statement appears to contain a typographical error.

## Question 4: $x - y$ is a factor of $x^n - y^n$

**Step 1: Base Case** – For $n = 1$: $x - y$ is a factor of $x - y$ ✓
**Step 2: Inductive Hypothesis** – Assume $x - y$ is a factor of $x^k - y^k$, i.e. $x^k - y^k = (x - y)Q(x)$ for some polynomial $Q(x)$.
**Step 3: Inductive Step** – Prove for $n = k + 1$:

$$\begin{aligned} x^{k+1} - y^{k+1} &= x^{k+1} - xy^k + xy^k - y^{k+1} \\ &= x(x^k - y^k) + y^k(x - y) \\ &= x(x - y)Q(x) + y^k(x - y) \\ &= (x - y)\left[xQ(x) + y^k\right] \end{aligned}$$

Therefore, $x - y$ is a factor of $x^n - y^n$ for all positive integers $n$. ✓

## Question 5: $n! > 2^{n-1}$ for $n \geq 2$

**Step 1: Base Case** – For $n = 2$: LHS $= 2! = 2$; RHS $= 2^{2-1} = 2$. Checking the originally stated form $n! > 2^n - 1$ for $n = 2$: $2 > 3$

is **false**, and for $n = 3$: $6 > 7$ is also false; it first holds at $n = 4$ $(24 > 15)$.

**Note**

The inequality as commonly intended is $n! > 2^{n-1}$ for $n \geq 2$ (verified: $2! = 2 \not> 2^1 = 2$, so equality holds at $n = 2$; strict inequality begins at $n = 3$: $3! = 6 > 2^2 = 4$). Care should be taken with the exact exponent when copying this statement, as a small change in exponent changes where the inequality first becomes true.

## Question 6: $4^n > 3^n + 2^n$ for $n \geq 2$

**Step 1: Base Case** – For $n = 2$: LHS $= 16$; RHS $= 9 + 4 = 13$; $16 > 13$ ✓
**Step 2: Inductive Hypothesis** – Assume $4^k > 3^k + 2^k$.
**Step 3: Inductive Step** – Prove for $n = k + 1$:

$$\begin{aligned} 4^{k+1} &= 4 \cdot 4^k > 4(3^k + 2^k) = 4 \cdot 3^k + 4 \cdot 2^k \\ &= 3 \cdot 3^k + 3^k + 2 \cdot 2^k + 2 \cdot 2^k \\ &= 3^{k+1} + 3^k + 2^{k+1} + 2^{k+1} > 3^{k+1} + 2^{k+1} \end{aligned}$$

Therefore, $4^n > 3^n + 2^n$ for all $n \geq 2$. ✓

## Question 7: $1 + nx \leq (1 + x)^n$ for $n \geq 2,\ x > -1$ (Bernoulli's Inequality)

**Step 1: Base Case** – For $n = 2$: LHS $= 1 + 2x$; RHS $= (1 + x)^2 = 1 + 2x + x^2$. Since $x^2 \geq 0$, $1 + 2x \leq 1 + 2x + x^2$ ✓
**Step 2: Inductive Hypothesis** – Assume $1 + kx \leq (1 + x)^k$.

**Step 3: Inductive Step** – Prove for $n = k + 1$, using $1 + x > 0$:

$$(1 + x)^{k+1} = (1 + x)(1 + x)^k \geq (1 + x)(1 + kx)$$
$$= 1 + (k + 1)x + kx^2 \geq 1 + (k + 1)x \quad (\text{since } kx^2 \geq 0)$$

Therefore, $1 + nx \leq (1 + x)^n$ for $n \geq 2$ and $x > -1$. ✓

## Question 8: Amount after $n$ years $= 1{,}000{,}000(1.06)^n$

**Step 1: Base Case** – For $n = 0$: Amount $= 1{,}000{,}000(1.06)^0 = 1{,}000{,}000$ ✓
**Step 2: Inductive Hypothesis** – Assume after $k$ years, amount $= 1{,}000{,}000(1.06)^k$.
**Step 3: Inductive Step** – After $(k + 1)$ years:

$$\text{Amount} = 1{,}000{,}000(1.06)^k \times 1.06 = 1{,}000{,}000(1.06)^{k+1}$$

Therefore, the formula holds for all $n$. ✓

## Question 9: $A(n) = P(1 + r)^n$ for all $n \geq 0$

**Step 1: Base Case** – For $n = 0$: $A(0) = P(1 + r)^0 = P$ ✓
**Step 2: Inductive Hypothesis** – Assume $A(k) = P(1 + r)^k$.
**Step 3: Inductive Step** – For $n = k + 1$:

$$A(k + 1) = A(k)(1 + r) = P(1 + r)^k(1 + r) = P(1 + r)^{k+1}$$

Therefore, the formula holds for all $n \geq 0$. ✓

## Question 10: Monthly savings increasing by Rs. 500

**Step 1:** Savings form an A.P.: $500, 1000, 1500, \ldots, 500n$. Total savings after $n$ months:

$$S_n = \frac{n}{2}(500 + 500n) = 250n(n+1)$$

**Step 2:** For $n = 24$:

$$S_{24} = 250 \times 24 \times 25 = 150{,}000$$

**Step 3:** Since $150{,}000 > 12{,}000$, his savings will comfortably reach at least Rs. 12,000.
**Answer:** Yes – after 24 months he will have Rs. 150,000. ✓

## Question 11: Loan balance $R_n = 2{,}000{,}000 - 50{,}000n$

**Step 1: Base Case** – For $n = 0$: $R_0 = 2{,}000{,}000 - 0 = 2{,}000{,}000$ ✓
**Step 2: Inductive Hypothesis** – Assume $R_k = 2{,}000{,}000 - 50{,}000k$.
**Step 3: Inductive Step** – For $n = k + 1$:

$$\begin{aligned} R_{k+1} &= R_k - 50{,}000 = 2{,}000{,}000 - 50{,}000k - 50{,}000 \\ &= 2{,}000{,}000 - 50{,}000(k+1) \end{aligned}$$

Therefore, the formula holds for all $n$. ✓

## Question 12: Savings formula $S(n)$

**Step 1:** Initial savings = Rs. 5,000; each month, add Rs. 1,000. After $n$ months: $S(n) = 5000 + 1000n$.
**Step 2: Base Case** – For $n = 0$: $S(0) = 5000 + 0 = 5000$ ✓
**Step 3: Inductive Hypothesis** – Assume $S(k) = 5000 + 1000k$.
**Step 4: Inductive Step** – For $n = k + 1$:

$$S(k+1) = S(k) + 1000 = 5000 + 1000k + 1000 = 5000 + 1000(k+1)$$

Therefore, $S(n) = 5000 + 1000n$ for all $n \geq 0$. ✓

# Summary of Results

## Overall Summary

| Question | Result |
|---|---|
| (vii) | Proven ✓ |
| (viii) | Proven ✓ |
| (ix) | Proven ✓ |
| (x) | Proven ✓ |
| (xi) | Proven ✓ |
| (xii) | Proven ✓ |
| 2(i) | Proven ✓ |
| 2(ii) | Proven ✓ |
| 2(iii) | Proven ✓ |
| 3 | Typo in statement |
| 4 | Proven ✓ |
| 5 | Statement needs correct exponent |
| 6 | Proven ✓ |
| 7 | Proven ✓ |
| 8 | Proven ✓ |
| 9 | Proven ✓ |
| 10 | Yes, reaches Rs. 150,000 |
| 11 | Proven ✓ |
| 12 | $S(n) = 5000 + 1000n$ |